Abstract

Eugenio Calabi claimed the following theorem at the International Congress of Mathematicians in 1954:

For any closed Kähler manifold (M^n, ω), for any $\Sigma \in 2\pi c_1(M)$, there is only one Kähler metric $\Omega \in [\omega]$, such that $Ric(\Omega) = \Sigma$.

This theorem can be reduced to solving the following complex Monge-Ampère equation w.r.t. some real-valued function φ:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F \omega^n,$$

where $F \in C^\infty(M; \mathbb{R})$.

Shing-Tung Yau proved this theorem in the remarkable paper *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*. As a corollary, for any compact Kähler manifold, with $c_1(M) \leq 0$, there is only one Kähler-Einstein metric.

This thesis will summarize Yau's original paper and try to provide details.

Keywords: Calabi Conjecture; Complex Monge-Ampère Equation; Kähler-Einstein Metric

Contents

Chapter 1

Preliminaries

Let M^n be a compact smooth manifold of real dimension n, and the Riemannian metric g be a smooth section defined on $T^*M \otimes T^*M$, and for any point $x \in M$, g is a positive definite quadratic form on T_xM. We know such a metric must exist. In the local coordinate $(x_1, x_2, \dots, x_n)$, the basis of TM is represented by $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\}$, then $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$, where g_{ij} is smooth function with respect to x:

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \tag{1.1}$$

We call (M^n, g) a Riemannian manifold.

Suppose there is a family of coordinate charts on the Riemannian manifold (M^{2m}, g), and the transition maps of these charts are biholomorphic. In that case, we can induce an automorphic map J on the tangent bundle TM of M, s.t. $J^2 = -\mathrm{id}$, which is known as the complex structure on M.

If we have the complex structure, we will call the J-invariant metric a Hermitian metric, s.t. $g(u, v) = g(Ju, Jv), \forall u, v \in TM$. Such metric must exist since the trivial metric on $\mathbb{C}^n$ is J-invariant. Take $\omega_g(u, v) = g(Ju, v), u, v \in TM$, then ω_g is a real $(1, 1)$-form, known as the fundamental form of complex manifold (M, g, J).

We can then $\mathbb{C}$-linearly extend this Riemannian metric g on $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$, whoes complex dimension equals to $2m$, and divide $T_{\mathbb{C}}M$ into the direct sum of $T^{(1,0)}M$ and $T^{(0,1)}M$, according to the eigen-subspace of the eigenvalues $\pm i$ of J. The complex dimension of $T^{(1,0)}M$ and $T^{(0,1)}M$ are both m.

In fact, take a complex coordinate $(z_1, z_2, \dots, z_m)$, where $z_i = x_i + \sqrt{-1}x_{m+i}$, we have:

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^{m+i}}, \quad J\frac{\partial}{\partial x^{m+i}} = -\frac{\partial}{\partial x^i}, \tag{1.2}$$

and

$$dz^i = dx^i + \sqrt{-1}dx^{m+i}, \quad d\bar{z}^i = dx^i - \sqrt{-1}dx^{m+i}. \tag{1.3}$$

Since for any smooth function f, the basic computational law should hold: $df = \frac{\partial f}{\partial x^i}dx^i = \frac{\partial f}{\partial z_i}dz^i + \frac{\partial f}{\partial \bar{z}^i}d\bar{z}^i$, thus for $i = 1, 2, \ldots, m$ we have:

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial x^{m+i}}\right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial x^{m+i}}\right). \tag{1.4}$$

Furthermore:

$$J\left(\frac{\partial}{\partial z^j}\right) = \sqrt{-1}\frac{\partial}{\partial z^j}, \quad J\left(\frac{\partial}{\partial \bar{z}^j}\right) = -\sqrt{-1}\frac{\partial}{\partial \bar{z}^j}. \tag{1.5}$$

In this way, a basis of $T^{(1,0)}M$ is $\{\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \ldots, \frac{\partial}{\partial z^m}\}$ a basis of $T^{(0,1)}M$ is $\{\frac{\partial}{\partial \bar{z}^1}, \frac{\partial}{\partial \bar{z}^2}, \ldots, \frac{\partial}{\partial \bar{z}^m}\}$.

Thus, by definition, for a Hermitian metric g, and for any i, j, we have:

$$g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) = 0. \tag{1.6}$$

Set $g_{i\bar{j}} := g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$, we can now write down the $\mathbb{C}$-linearly extended J-invariant metric in local coordinate:

$$g = \sum_{i,j}\left(g_{i\bar{j}}dz^i \otimes d\bar{z}^j + \bar{g}_{i\bar{j}}d\bar{z}^i \otimes dz^j\right). \tag{1.7}$$

Note that the second part of the above equation can be obtained by taking the conjugate transpose of the first part. This explains why we call the J-invariant metric also the Hermitian metric. And we usually take the first part, i.e., $g = \sum_{i,j} g_{i\bar{j}}dz^i \otimes d\bar{z}^j$ as the metric of a complex manifold M.

Definition 1. *We call a triple $(M^m, g_{\mathbb{C}}, J)$ satisfies the above conditions a complex manifold, with complex dimension m. Furthermore, if only consider the Riemannian manifold (M, g) and its Levi-Civita connection ∇ satisfies $\nabla J = 0$, we call (M^m, g, J) a Kähler manifold, $\sum_{i,j} g_{i\bar{j}}dz^i \otimes d\bar{z}^j$ the Kähler metric, and $\omega_g = \sqrt{-1}\sum_{i,j} g_{i\bar{j}}dz^i \wedge d\bar{z}^j$ the Kähler form.*

Remark 1. *We will only use g to represent the Kähler metric instead of $g_{\mathbb{C}}$. Since they are equivalent, we always mix and use the names of the Kähler metric and Kähler form.*

Actually, we have the following equivalent conditions to see if a complex manifold is Kähler[Huy05]:

Lemma 1. *For a complex manifold (M, g, J), the followings are equivalent*

(i) (M, g, J) *is Kähler;*

(ii) $\nabla J = 0$*;*

(iii) $d\omega_g = 0$;

(iv) *For any* i, j, k, *we have:*

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i} \text{ and } \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}. \tag{1.8}$$

(v) *For any point* $p \in M$, *there is a local holomorphic normal coordinate, i.e., exists a local holomorphic coordinate* $(z_1, \ldots, z_m)$, *s.t. for all* i, j, *we have:*

$$z_i(p) = 0, g_{i\bar{j}} = \delta_{ij}, dg_{i\bar{j}} = 0. \tag{1.9}$$

Next, we compute the curvature on a Kähler manifold in local coordinates. The reference is [Tia00]:

We take a local coordinate $(z_1, \ldots, z_n)$ of M, and the basis of the tangent space $T_{\mathbb{C}}M$ is $\left(\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\right)$. First, we compute the Christoffel symbol Γ^k_{ij}:

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} := \Gamma^k_{ij} \frac{\partial}{\partial z^k} + \Gamma^{\bar{k}}_{ij} \frac{\partial}{\partial \bar{z}^k}, \tag{1.10}$$

and

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}^j} := \Gamma^k_{i\bar{j}} \frac{\partial}{\partial z^k} + \Gamma^{\bar{k}}_{i\bar{j}} \frac{\partial}{\partial \bar{z}^k}. \tag{1.11}$$

Due to the Kähler condition $\nabla J = 0$, and the complex structure satisfies $J\frac{\partial}{\partial z^i} = \sqrt{-1}\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial \bar{z}^i} = -\sqrt{-1}\frac{\partial}{\partial \bar{z}^i}$, we have

$$\nabla_{\frac{\partial}{\partial z^i}} \left(J \frac{\partial}{\partial z^j} \right) = J \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} + \left(\nabla_{\frac{\partial}{\partial z^i}} J \right) \frac{\partial}{\partial z^j}. \tag{1.12}$$

The second term on the right-hand side is equal to 0. According to the definition of the Christoffel symbol, we immediately have:

$$\sqrt{-1} \left(\Gamma^k_{ij} \frac{\partial}{\partial z^k} + \Gamma^{\bar{k}}_{ij} \frac{\partial}{\partial \bar{z}^k} \right) = J \left(\Gamma^k_{ij} \frac{\partial}{\partial z^k} + \Gamma^{\bar{k}}_{ij} \frac{\partial}{\partial \bar{z}^k} \right) = \sqrt{-1} \left(\Gamma^k_{ij} \frac{\partial}{\partial z^k} - \Gamma^{\bar{k}}_{ij} \frac{\partial}{\partial \bar{z}^k} \right). \tag{1.13}$$

Rewrite the above equation, we get $\Gamma^{\bar{k}}_{ij} = 0$. Similarly, we can compute $\nabla_{\frac{\partial}{\partial z^i}} \left(J\frac{\partial}{\partial \bar{z}^j} \right)$, and get $\Gamma^{\bar{k}}_{i\bar{j}} = \Gamma^k_{i\bar{j}} = 0$, thus the left non-zero Christoffel symbols are Γ^k_{ij} and its conjugate $\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}}$. Now, let's compute Γ^k_{ij}

$$\frac{\partial g_{j\bar{k}}}{\partial z^i} = \frac{\partial}{\partial z^i} g \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k} \right) = g \left(\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k} \right) = g \left(\Gamma^l_{ij} \frac{\partial}{\partial z^l}, \frac{\partial}{\partial \bar{z}^k} \right) = \Gamma^l_{ij} g_{l\bar{k}}. \tag{1.14}$$

If we set $(g^{l\bar{k}})$ to be the transpose of inverse matrix of $(g_{l\bar{k}})$, then:

$$\Gamma^l_{ij} = g^{l\bar{k}} \frac{\partial g_{j\bar{k}}}{\partial z^i} = g^{l\bar{k}} \frac{\partial g_{i\bar{k}}}{\partial z^j}. \tag{1.15}$$

Next, we want to $\mathbb{C}$-linearly extend the real curvature tensor For any $u, v, w \in T_{\mathbb{C}}M$, define the curvature $R(u,v)w := \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$, where $[u,v]$ is the Lie bracket of vectors u, v, and equals to 0 in local coordinate. Furthermore, in local coordinate, we set:

$$R_{i\bar{j}k\bar{l}} := \left\langle R\left(\partial_i, \partial_{\bar{j}}\right)\partial_k, \partial_{\bar{l}}\right\rangle. \tag{1.16}$$

Actually, we can compute $R_{i\bar{j}k\bar{l}}$ explicitly:

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= \left\langle\left(\nabla_i\nabla_{\bar{j}} - \nabla_{\bar{j}}\nabla_i\right)\partial_k, \partial_{\bar{l}}\right\rangle = -\left\langle\nabla_{\bar{j}}\left(\Gamma^p_{ik}\partial_p\right), \partial_{\bar{l}}\right\rangle \\
&= -\partial_{\bar{j}}\Gamma^p_{ik}g_{p\bar{l}} = -\partial_{\bar{j}}\left(g^{\bar{q}p}\frac{\partial g_{k\bar{q}}}{\partial z^i}\right)g_{p\bar{l}} \\
&= -g^{\bar{q}p}\frac{\partial^2 g_{k\bar{q}}}{\partial z^i\partial\bar{z}^j}g_{p\bar{l}} + g^{\bar{q}s}g^{\bar{t}p}\frac{\partial g_{s\bar{t}}}{\partial\bar{z}^j}\frac{\partial g_{k\bar{q}}}{\partial z^i}g_{p\bar{l}} \\
&= -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i\partial\bar{z}^j} + g^{\bar{q}s}\frac{\partial g_{s\bar{l}}}{\partial\bar{z}^j}\frac{\partial g_{k\bar{q}}}{\partial z^i}.
\end{aligned} \tag{1.17}$$

From the above computation and the equivalent condition of the Kähler manifold, we have $R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}}$, i.e., the complex curvature operator has more symmetry properties compares to the real one.

By definition, we find:

$$R(Ju,Jv,w,x) = R(w,x,Ju,Jv) = g(R(w,x)Ju,Jv) = g(R(w,x)u,v) = R(u,v,w,x). \tag{1.18}$$

Thus, we can define the Ricci curvature tensor:

$$Rc := R_{i\bar{j}}dz^i \otimes d\bar{z}^j = g^{k\bar{l}}R_{i\bar{j}k\bar{l}}dz^i \otimes d\bar{z}^j. \tag{1.19}$$

And from 1.18, we know the Ricci tensor is J-invariant. Similarly to the definition of fundamental form, we have:

$$Ric(\omega_g) := Rc(J\cdot,\cdot) = \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j. \tag{1.20}$$

Thus, we find $Ric(\omega_g)$ to be a real closed $(1,1)$-form, known as the Ricci form of the complex manifold (M,g,J). We have:

$$R_{i\bar{j}} = Rc\left(\partial_i, \partial_{\bar{j}}\right) = g^{\bar{l}k}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z^i\partial\bar{z}^j}\log\det\left(g_{p\bar{q}}\right). \tag{1.21}$$

Therefore, we also have $Ric(\omega_g) = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$.

Then, we write down a useful lemma. The proofs of the following lemma and proposition can be found in [Tia00], Chapter 2:

Lemma 2 ($\partial\bar{\partial}$-Lemma)**.** *Suppose (M,g,J) is a compact Kähler manifold. If $\alpha \in H^{(1,1)}_{\bar{\partial}}(M,\mathbb{C})$ is d-exact, then there is smooth function β, s.t. $\alpha = \partial\bar{\partial}\beta$. Furthermore, if α is a real form, we can always choose $\beta \in C^\infty(M;\mathbb{R})$, s.t. $\alpha = \sqrt{-1}\partial\bar{\partial}\beta$.*

The following proposition of Chern class is important:

Proposition 1. *Given a Kähler manifold (M, g, J), whoes Kähler form is ω_g, we can define an $(1,1)$-form*

$$\Omega := g^{j\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l, \tag{1.22}$$

and set:

$$\det\left(I + \frac{t\sqrt{-1}}{2\pi}\Omega\right) = 1 + t\phi_1(g) + t^2\phi_2(g) + \cdots \tag{1.23}$$

Then we have:

(i) $d\phi_i(g) = 0$*, thus* $[\phi_i] \in H^{i,i}_{\bar{\partial}}(M, \mathbb{C}) \cap H^{2i}(M, \mathbb{R})$*;*

(ii) $[\phi_i(g)]$ *is independent of the choice of the metric* g*. It only depends on the complex structure* J *of the manifold;*

(iii) The i^{th} *Chern class of the manifold is represented by* $c_i(M) := [\phi_i(g)]$*.*

In particular, we can now compute the representative element of the first Chern class of a complex manifold:

$$\phi_1(g) = \frac{\sqrt{-1}}{2\pi}\Omega^i_i = \frac{\sqrt{-1}}{2\pi} R_{k\bar{l}} dz^k \wedge d\bar{z}^l = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(g_{i\bar{j}}). \tag{1.24}$$

Eugenio Calabi claimed in 1954[Cal54] that given an arbitrary element in the first Chern class of a complex manifold, we can always find a Kähler metric, s.t. its Ricci form is exactly the given element. Calabi proved the uniqueness by himself, i.e., in every Kähler class (the cohomology class of Kähler metric), at most, one Kähler metric satisfies the condition.

In 1978, Shing-Tung Yau confirmed this statement and proved that there is exactly one solution in the Kähler class of the initial Kähler metric[Yau78]. Furthermore, he proved the existence of the Kähler-Einstein metric on the compact Kähler manifold, whose first Chern class is non-positive. Specifically, there is always a Ricci-flat metric on the compact Kähler manifold whose first Chern class vanishes. This is known to be the Calabi-Yau manifold.

We write the above statement mathematically:

For a compact Kähler manifold (M^m, ω), if we can fine a Kähler metric $\sqrt{-1}\sum_{i,j}\tilde{g}_{i\bar{j}} dz^i \wedge d\bar{z}^j$ which is cohomologous to the initial Kähler metric $\sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$, and the cohomologous class of its Ricci form

$$Ric(\omega_{\tilde{g}}) = -\sqrt{-1}\partial\bar{\partial}\log\det(\tilde{g}_{i\bar{j}})$$

in $H^{1,1}_{\bar{\partial}}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$ is $2\pi c_1(M)$.

We know that $2\pi c_1(M)$ can be represented by $-\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$, and since the difference of these two elements is $\bar{\partial}$-closed and d-exact, by the $\partial\bar{\partial}$-Lemma, we know there is smooth real-valued function $F \in C^\infty(M; \mathbb{R})$, s.t.:

$$-\sqrt{-1}\partial\bar{\partial}\log\det(\tilde{g}_{i\bar{j}}) = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}) + \sqrt{-1}\partial\bar{\partial}F. \tag{1.25}$$

i.e.:

$$\partial\bar{\partial}\log\frac{\det(\tilde{g}_{i\bar{j}})}{\det(g_{i\bar{j}})} = \partial\bar{\partial}F. \tag{1.26}$$

Since the manifold M is compact, and $\det(\tilde{g}_{i\bar{j}})/\det(g_{i\bar{j}})$ is globally defined (it's a ratio, and if we change the local coordinate, the extra terms will be canceled), by the maximal principle, we know that $\log\frac{\det(\tilde{g}_{i\bar{j}})}{\det(g_{i\bar{j}})} - F$ is a constant. Thus, there is a positive constant C, s.t.:

$$\det(\tilde{g}_{i\bar{j}}) = C\exp\{F\}\det(g_{s\bar{t}}). \tag{1.27}$$

Again, the above two metrics are supposed in the same Kähler class, by $\partial\bar{\partial}$-Lemma, there is a real-valued smooth function $\varphi \in C^\infty(M;\mathbb{R})$, satisfies:

$$\sqrt{-1}\sum_{i,j}\tilde{g}_{i\bar{j}}dz^i \wedge d\bar{z}^j = \sqrt{-1}\sum_{i,j}g_{i\bar{j}}dz^i \wedge d\bar{z}^j + \sqrt{-1}\partial\bar{\partial}\varphi. \tag{1.28}$$

i.e.:

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial\bar{z}^j}. \tag{1.29}$$

Therefore, the question Calabi asked can be deduced into solving the following complex Monge-Ampère equation of the real-valued function φ:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right) = C\exp\{F\}\det\left(g_{i\bar{j}}\right). \tag{1.30}$$

Chapter 2

A Priori Estimates of the Complex Monge-Ampère Equation

Suppose M is a compact Kähler manifold, we will study the following complex Monge-Ampère equation of the real-valued function φ:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) = \exp\{F\} \det(g_{i\bar{j}}), \tag{2.1}$$

where we assume $F \in C^3(M;\mathbb{R}), \varphi \in C^5(M;\mathbb{R})$.

According to the assumption of the conjecture, the left-hand side of the above equation is the determinant of the new Kähler metric, thus we need to assume φ should satisfy that:

$$g' := \sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j})dz^i \otimes d\bar{z}^j \tag{2.2}$$

is a positive definite Hermitian metric, and in the same Kähler class as the initial metric $g = \sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$.

Our goal is to show the existence of the demanding smooth solution to the equation 2.1. For the complex Monge-Ampère equation, if we differentiate both sides i.e. linearize, we will get a uniformly elliptic equation. Thus, we can use the classic methods in solving the elliptic equations. First of all, we need to get the a priori estimates of φ and show the existence of a smooth solution by the continuity method.

2.1 A Priori C^2 Estimates

We can see from 2.1 that if we add a constant to a solution φ, it still will be a solution. Thus, we can assume φ satisfies:

$$\int_M \varphi = 0, \tag{2.3}$$

To achieve this, we only need to substitute the previous φ with $\varphi - \frac{1}{\mathrm{vol}(M)} \int_M \varphi$.

Take natural logarithm on both sides of 2.1, and differentiate w.r.t. $\frac{\partial}{\partial z^k}$. Using the following formula to differentiate a determinant:

$$\frac{\partial}{\partial z^k} \det(g_{i\bar{j}}) = \det(g_{i\bar{j}}) g^{p\bar{q}} \frac{\partial g_{p\bar{q}}}{\partial z^k}, \tag{2.4}$$

We have:

$$\begin{aligned}\frac{\partial}{\partial z^k}\left(\log\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) - \log\det(g_{i\bar{j}})\right) &= g'^{i\bar{j}}\left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3\varphi}{\partial z^i\partial\bar{z}^j\partial z^k}\right) - g'^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial z^k} \\ &= \frac{\partial F}{\partial z^k}.\end{aligned} \tag{2.5}$$

where $g^{i\bar{j}}$ represents the (i,j)-th element in the transpose of the inverse matrix of $(g_{i\bar{j}})$.

Again, differentiate both sides of the above equation w.r.t. $\frac{\partial}{\partial \bar{z}^l}$. Using the following elementary formula to differentiate the inverse element of a matrix:

$$\frac{\partial g^{i\bar{j}}}{\partial z^k} = -g^{i\bar{q}} \frac{\partial g_{p\bar{q}}}{\partial z^k} g^{p\bar{j}}, \tag{2.6}$$

we have:

$$\begin{aligned}&-\sum_{i,j,t,n} g'^{t\bar{j}} g'^{i\bar{n}} \left(\frac{\partial g_{t\bar{n}}}{\partial \bar{z}^l} + \frac{\partial^3\varphi}{\partial z^t \partial \bar{z}^n \partial \bar{z}^l}\right)\left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3\varphi}{\partial z^i\partial\bar{z}^j\partial z^k}\right) \\ &\qquad + \sum_{i,j} g'^{i\bar{j}}\left(\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \frac{\partial^4\varphi}{\partial z^i\partial\bar{z}^j\partial z^k\partial\bar{z}^l}\right) \\ &\qquad + \sum_{i,j,k,n} g^{t\bar{j}} g^{i\bar{n}} \frac{\partial g_{t\bar{n}}}{\partial \bar{z}^l}\frac{\partial g_{i\bar{j}}}{\partial z^k} - \sum_{i,j} g^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z^k\partial\bar{z}^l} = \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l}.\end{aligned} \tag{2.7}$$

Let Δ be the Laplacian associated with the metric $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Since the Kähler metric comes from $\mathbb{C}$-linearly extending of the real J-invariant metric. In this case, the metric can be written in a two-conjugate-block diagonal matrix. Thus, the determinant of the Kähler metric $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$ equals the arithmetic square root of the real Riemannian metric tensor.

Recall the Laplace-Beltrami formula:

$$\Delta = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^i}\left(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial x^j}\right). \tag{2.8}$$

Apply the above formula to the Kähler manifold, we have:

$$\Delta = \frac{1}{\det(g_{i\bar{j}})} \frac{\partial}{\partial z^i}\left(\det(g_{i\bar{j}}) g^{i\bar{j}} \frac{\partial}{\partial \bar{z}^j}\right) = g^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}. \tag{2.9}$$

We always denote Δ as the Laplacian associated with the Kähler metric $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$, and Δ' as the Laplacian associated with $\sum_{i,j}\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) dz^i \otimes d\bar{z}^j$. Thus, by using the differentiation formula for the inverse element of the matrix:

$$\begin{aligned}
\Delta'(\Delta\varphi) &= \sum_{k,l} g'^{k\bar{l}} \frac{\partial^2}{\partial z^k \partial \bar{z}^l}\left(g^{i\bar{j}} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \\
&= \sum_{i,j,k,l} g'^{k\bar{l}} g^{i\bar{j}} \frac{\partial^4 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l} + \sum_{i,j,k,l} g'^{k\bar{l}} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \\
&+ \sum_{i,j,k,l} g'^{k\bar{l}} \frac{\partial g^{i\bar{j}}}{\partial z^k} \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial \bar{z}^l} + \sum_{i,j,k,l} g'^{k\bar{l}} \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^l} \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k}.
\end{aligned} \tag{2.10}$$

Note that the left-hand side of the above equation is independent of the choice of local coordinates. We can conveniently use the normal coordinate to compute the right-hand side, and the Kähler condition also ensures we can take the normal coordinate. Thus, at a certain point, we have: $g_{i\bar{j}} = \delta_{ij}, \frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^l} = 0$. Then, we can see the third and fourth terms of the right-hand side equal to 0. Furthermore, we can get the fourth order term of φ from 2.7, and substitute it into the above equation:

$$\Delta'(\Delta\varphi) = \Delta F + \sum_{i,j,k,n,l} g'^{k\bar{j}} g'^{i\bar{n}} \varphi_{k\bar{n}\bar{l}} \varphi_{i\bar{j}l} + \sum_{i,j,l} g'^{i\bar{j}} R_{i\bar{j}l\bar{l}} - \sum_{i,l} R_{i\bar{i}l\bar{l}} + \sum_{i,j,k,l} g'^{k\bar{l}} R_{i\bar{j}k\bar{l}} \varphi_{j\bar{i}}. \tag{2.11}$$

For the same reason, we can choose another special coordinate, s.t.:

$$g_{i\bar{j}} = \delta_{ij}, \varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{i}}. \tag{2.12}$$

This coordinate always exists because we can first choose the coordinate, diagonalizing the metric matrix $(g_{i\bar{j}})$, then use the unitary matrix Q, in the sense of $A \to QA\bar{Q}^T$, to diagonalize the matrix $A = (\varphi_{i\bar{j}})$.

Therefore,

$$g'^{i\bar{j}} = \frac{\delta_{ij}}{1 + \varphi_{i\bar{i}}}, \tag{2.13}$$

In this special coordinate, we can now simplify the last three terms in 2.11:

$$
\begin{aligned}
&\sum_{i,j,l} g'^{i\bar{j}} R_{i\bar{j}l\bar{l}} - \sum_{i,l} R_{i\bar{i}l\bar{l}} + \sum_{i,j,k,l} g'^{k\bar{l}} R_{i\bar{j}k\bar{l}} \varphi_{i\bar{j}} \\
&= -\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}} + \sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} \\
&= -\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\varphi_{i\bar{i}}\left(\varphi_{l\bar{l}}-\varphi_{i\bar{i}}\right)}{\left(1+\varphi_{i\bar{i}}\right)\left(1+\varphi_{l\bar{l}}\right)} \\
&= \frac{1}{2}\left\{-\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\varphi_{i\bar{i}}\left(\varphi_{l\bar{l}}-\varphi_{i\bar{i}}\right)}{\left(1+\varphi_{i\bar{i}}\right)\left(1+\varphi_{l\bar{l}}\right)} - \sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\varphi_{l\bar{l}}\left(\varphi_{i\bar{i}}-\varphi_{l\bar{l}}\right)}{\left(1+\varphi_{i\bar{i}}\right)\left(1+\varphi_{l\bar{l}}\right)}\right\} \\
&= \frac{1}{2}\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\left(\varphi_{l\bar{l}}-\varphi_{i\bar{i}}\right)^2}{\left(1+\varphi_{i\bar{i}}\right)\left(1+\varphi_{l\bar{l}}\right)}.
\end{aligned}
\tag{2.14}
$$

Note that the third equality changed the position of i, j and added together. Thus there is an $\frac{1}{2}$ coefficient. Doing so is to obtain the square term in the last equality. Also, when $i = l$, the term multiplies by $R_{i\bar{i}l\bar{l}}$ equals to 0.

We continue to simplify the above equation:

$$
\begin{aligned}
\frac{1}{2}\sum_{i,l} \frac{\left(\varphi_{l\bar{l}}-\varphi_{i\bar{i}}\right)^2}{\left(1+\varphi_{i\bar{i}}\right)\left(1+\varphi_{l\bar{l}}\right)} &= \frac{1}{2}\sum_{i,l} \frac{\left[\left(1+\varphi_{l\bar{l}}\right)-\left(1+\varphi_{i\bar{i}}\right)\right]^2}{\left(1+\varphi_{i\bar{i}}\right)\left(1+\varphi_{l\bar{l}}\right)} \\
&= \frac{1}{2}\sum_{i,l}\left(\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} + \frac{1+\varphi_{l\bar{l}}}{1+\varphi_{i\bar{i}}} - 2\right) \\
&= \sum_{i,j} \frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} - m^2.
\end{aligned}
\tag{2.15}
$$

Since the sum takes over arbitrary i, j, it adds m^2 times in total. We can see the last equality holds.

Then, substitute 2.14 and 2.15 into 2.11, we get the inequality that is useful later:

$$
\Delta'(\Delta\varphi) \geq \Delta F + \sum_{l} g'^{k\bar{j}} g'^{i\bar{n}} \varphi_{k\bar{n}\bar{l}} \varphi_{i\bar{j}l} + \left(\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\left[\sum_{i,l} \frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} - m^2\right]. \tag{2.16}
$$

The above inequality sheds some light on how to get the second order a priori estimates of φ. We can start by considering the normalized Laplacian Δ' of some function w.r.t. $\Delta\varphi$. At the maximal point, the normalized Laplacian is non-positive, and we hope to get the C^2 estimates.

Next, we have the following straightforward computation in the special coordinate:

$$
\Delta'\varphi = g'^{i\bar{j}} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} = \frac{\delta_{ij}}{1+\varphi_{i\bar{i}}} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} = \sum_i \frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}} = m - \sum_i \frac{1}{1+\varphi_{i\bar{i}}}. \tag{2.17}
$$

In Yau's paper[Yau78], he considered the function $\exp\{-C\varphi\}(m+\Delta\varphi)$, where C is a positive constant.

During the process of computing $\Delta'(\exp\{-C\varphi\}(m+\Delta\varphi))$, the third and fourth order terms of φ will occur. By 2.16, the lower bound of the fourth-order terms can be controlled by the third-order terms, and we will see in the computation that the extra third-order terms in this step can be eliminated with the original third-order terms. End up having the upper bound of C^2 estimates. These estimates will rely on the C^0 estimates of φ. Therefore, if we can also prove the C^0 estimates, together with the equation 2.1, by the mean value inequality, we will get uniform C^2 estimates of φ.

Now, let's do the computation:

$$\begin{aligned}\Delta'(\exp\{-C\varphi\}(m+\Delta\varphi)) &= g'^{i\bar{j}}\frac{\partial^2}{\partial z^i\partial\bar{z}^j}(\exp\{-C\varphi\}(m+\Delta\varphi))\\ &= C^2\exp\{-C\varphi\}\left(\sum_{i,j}g'^{i\bar{j}}\varphi_i\varphi_{\bar{j}}\right)(m+\Delta\varphi)\\ &- C\exp\{-C\varphi\}\sum_{i,j}g'^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}\\ &- C\exp\{-C\varphi\}\sum_{i,j}g'^{i\bar{j}}(\Delta\varphi)_i\varphi_{\bar{j}}\\ &- C\exp\{-C\varphi\}\Delta'\varphi(m+\Delta\varphi)+\exp\{-C\varphi\}\Delta'(\Delta\varphi).\end{aligned} \tag{2.18}$$

For the complex inner product, we still have the following Cauchy-Schwartz inequality, which plays an important role in our computation:

$$\langle\nabla\varphi,\nabla\Delta\varphi\rangle_{\mathbb{C}} \le |\nabla\varphi|\cdot|\nabla\Delta\varphi|, \tag{2.19}$$

where $\nabla\varphi := g^{i\bar{j}}\frac{\partial\varphi}{\partial\bar{z}^j}\frac{\partial}{\partial z^i}$. We can see that the complex gradient is only half of the classic gradient in Riemannian geometry.

Using the Cauchy-Schwartz inequality to the second and third-order terms in the right-hand side of 2.18:

$$\sum_{i,j}g'^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}} \le \sqrt{g'^{i\bar{j}}\varphi_i\varphi_{\bar{j}}}\sqrt{g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_i}, \tag{2.20}$$

Similarly,

$$\sum_{i,j}g'^{i\bar{j}}(\Delta\varphi)_i\varphi_{\bar{j}} \le \sqrt{g'^{i\bar{j}}\varphi_i\varphi_{\bar{j}}}\sqrt{g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_i}, \tag{2.21}$$

add the above two equations together:

$$\sum_{i,j}g'^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}+\sum_{i,j}g'^{i\bar{j}}(\Delta\varphi)_i\varphi_{\bar{j}} \le 2\sqrt{g'^{i\bar{j}}\varphi_i\varphi_{\bar{j}}}\sqrt{g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_i}. \tag{2.22}$$

To eliminate the first term in the right-hand side of 2.18, we use the following Cauchy-Schwartz inequality with ε:

$$2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}, \varepsilon > 0. \tag{2.23}$$

Consider the right-hand side of 2.22, using the above inequality, and take

$$a = \sqrt{g'^{i\bar{j}}\varphi_i\varphi_{\bar{j}}}, b = \sqrt{g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_i}, \varepsilon = C(m+\Delta\varphi),$$

we have:

$$\begin{aligned}
&- C\exp\{-C\varphi\}\sum_{i,j} g'^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}} - C\exp\{-C\varphi\}\sum_{i,j} g'^{i\bar{j}}(\Delta\varphi)_i\varphi_{\bar{j}} \\
&\geq -C\exp\{-C\varphi\}\left(C(m+\Delta\varphi)g'^{i\bar{j}}\varphi_i\varphi_{\bar{j}} + \frac{g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_i}{C(m+\Delta\varphi)}\right) \\
&= -C^2\exp\{-C\varphi\}\left(g'^{i\bar{j}}\varphi_i\varphi_{\bar{j}}\right)(m+\Delta\varphi) - (m+\Delta\varphi)^{-1}\exp\{-C\varphi\}\sum_{k,j} g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}.
\end{aligned} \tag{2.24}$$

Substitute the above inequality into 2.18, we find:

$$\begin{aligned}
\Delta'(\exp\{-C\varphi\}(m+\Delta\varphi)) \geq &- (m+\Delta\varphi)^{-1}\exp\{-C\varphi\}\sum_{k,j} g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \\
&- C\exp\{-C\varphi\}\left(\Delta'\varphi\right)(m+\Delta\varphi) + \exp\{-C\varphi\}\Delta'(\Delta\varphi).
\end{aligned} \tag{2.25}$$

This preparation shows some symmetric properties of the third-order terms of φ. Remember that by 2.16, the last term in the above inequality relates to the second and third-order terms of φ. Therefore, we will try to eliminate the third-order terms on the right-hand side. This is a tricky step.

As before, in our special coordinate, we have $g'^{i\bar{j}} = \frac{\delta_{ij}}{1+\varphi_{i\bar{i}}}$, and also, $\Delta\varphi$ is a real-valued function, we have:

$$\sum_i (\Delta\varphi)_i \cdot (\Delta\varphi)_{\bar{i}} = \sum_i |(\Delta\varphi)_i|^2 = \sum_i \left|\sum_k \varphi_{k\bar{k}i}\right|^2. \tag{2.26}$$

Considering the third order terms of φ in the right-hand side of 2.25, i.e. the first term and the last term, and plug 2.16 and 2.26 into it:

$$\begin{aligned}
-(m+\Delta\varphi)^{-1}&\sum_{i,i} g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} + \Delta'(\Delta\varphi) \\
&\geq - (m+\Delta\varphi)^{-1}\sum_i (1+\varphi_{i\bar{i}})^{-1}\left|\sum_k \varphi_{k\bar{k}i}\right|^2 \\
&\quad + \Delta F + \sum_{i,j,k} (1+\varphi_{k\bar{k}})^{-1}(1+\varphi_{i\bar{i}})^{-1}\varphi_{k\bar{i}\bar{j}}\varphi_{i\bar{k}\bar{j}} \\
&\quad + \sum_{i,l}\left(\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\left[\sum_{i,l}\frac{1+\varphi_{l\bar{l}}}{1+\varphi_{i\bar{i}}} - m^2\right].
\end{aligned} \tag{2.27}$$

We consider the first term on the right-hand side, write an extra term $(1+\varphi_{k\bar{k}})^{1/2}$ both in the numerator and denominator and use the Cauchy-Schwartz inequality:

$$
\begin{aligned}
&(m+\Delta\varphi)^{-1}\sum_i (1+\varphi_{i\bar{i}})^{-1}\left|\sum_k \varphi_{k\bar{k}i}\right|^2 \\
&\qquad= (m+\Delta\varphi)^{-1}\sum_i (1+\varphi_{i\bar{i}})^{-1}\left|\sum_k \frac{\varphi_{k\bar{k}i}}{(1+\varphi_{k\bar{k}})^{1/2}}(1+\varphi_{k\bar{k}})^{1/2}\right|^2 \\
&\qquad\le (m+\Delta\varphi)^{-1}\left(\sum_{i,k}(1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{k\bar{k}})^{-1}\varphi_{k\bar{k}i}\varphi_{\bar{k}k\bar{i}}\right)\left(\sum_k (1+\varphi_{k\bar{k}})\right) \\
&\qquad= \sum_{i,k}(1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{k\bar{k}})^{-1}\varphi_{k\bar{k}i}\varphi_{\bar{k}k\bar{i}} \\
&\qquad= \sum_{i,k}(1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{k\bar{k}})^{-1}\varphi_{i\bar{k}k}\varphi_{k\bar{i}\bar{k}} \\
&\qquad\le \sum_{i,j,k}(1+\varphi_{k\bar{k}})^{-1}(1+\varphi_{i\bar{i}})^{-1}\varphi_{i\bar{k}j}\varphi_{k\bar{i}\bar{j}}.
\end{aligned}
\tag{2.28}
$$

Because we have $\sum_k(1+\varphi_{k\bar{k}}) = m+\sum_k \varphi_{k\bar{k}} = m+\Delta\varphi$, the fourth line in the above computation holds.

Next, we need to use the commutation formula for covariant derivatives up to the third order. The basic rule is that we can always interchange the first and second slots.

As for the second and third slots, if one indice is barred and the other is not, by the Ricci identity, curvature terms show up after switching these slots. Otherwise, we can also switch them freely, i.e., we are using the following equations:

$$\varphi_{k\bar{k}i} = \varphi_{\bar{k}ki} = \varphi_{\bar{k}ik} = \varphi_{i\bar{k}k}, \tag{2.29}$$

and

$$\varphi_{\bar{k}k\bar{i}} = \varphi_{k\bar{k}\bar{i}} = \varphi_{k\bar{i}\bar{k}}. \tag{2.30}$$

Plug the above two equations 2.29 and 2.30 into 2.28, we see the fifth line is valid. In the last line, we substitute the summable indice k with k, j, and take the sum, respectively. According to the positivity of the inner product, we know that inequality holds.

Substitute 2.28 into the inequality 2.27. We see that all the third-order terms on the right-hand side are eliminated, only the second-order terms left w.r.t. φ:

$$
\begin{aligned}
-(m+\Delta\varphi)^{-1}&\sum_{i,j} g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} + \Delta'(\Delta\varphi) \\
&\ge \Delta F + \left(\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\left[\sum_{i,l}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} - m^2\right].
\end{aligned}
\tag{2.31}
$$

Then, we substitute the above inequality 2.31 into 2.25:

$$\Delta'(\exp\{-C\varphi\}(m+\Delta\varphi)) \geq \exp\{-C\varphi\}\left[\Delta F + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\left(\sum_{i,l}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} - m^2\right)\right] \\ - C\exp\{-C\varphi\}\left(\Delta'\varphi\right)(m+\Delta\varphi). \tag{2.32}$$

Continue to simplify the above inequality:

Since at the maximal point of $\exp\{-C\varphi\}(m+\Delta\varphi)$, the left-hand side of 2.32 is smaller or equal to 0, we will try to deduce the right-hand side to terms only related to $\Delta\varphi$, and thus get upper estimates of the second terms.

In the special coordinate, we have:

$$\Delta'\varphi = m - \sum_i \frac{1}{1+\varphi_{i\bar{i}}}.$$

We also have the following identity:

$$\sum_{i,l}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}} = \left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right)\left(\sum_l (1+\varphi_{l\bar{l}})\right) = (m+\Delta\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right), \tag{2.33}$$

Substitute the above equation into 2.32:

$$\begin{aligned}\Delta'(\exp\{-C\varphi\}(m+\Delta\varphi)) &\geq \exp\{-C\varphi\}\left(\Delta F - m^2\inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right) \\ &+ \exp\{-C\varphi\}\inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\left(\sum_{i,l}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{l\bar{l}}}\right) \\ &- C\exp\{-C\varphi\}m(m+\Delta\varphi) + C\exp\{-C\varphi\}(m+\Delta\varphi)\sum_i\frac{1}{1+\varphi_{i\bar{i}}} \\ &= \exp\{-C\varphi\}\left(\Delta F - m^2\inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right) - C\exp\{-C\varphi\}m(m+\Delta\varphi) \\ &+ \left(C + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right)\exp\{-C\varphi\}(m+\Delta\varphi)\left(\sum_i\frac{1}{1+\varphi_{i\bar{i}}}\right).\end{aligned} \tag{2.34}$$

We want to represent the term $\sum_i \frac{1}{1+\varphi_{i\bar{i}}}$ in the right-hand side by $m+\Delta\varphi$, thus the following inequality is needed:

$$\sum_i \frac{1}{1+\varphi_{i\bar{i}}} \geq \left(\frac{\sum_i (1+\varphi_{i\bar{i}})}{\prod_i (1+\varphi_{i\bar{i}})}\right)^{1/(m-1)}. \tag{2.35}$$

We first prove the above inequality. Take the $(m-1)$-th power on both sides:

$$\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right)^{(m-1)} \geq \frac{\sum_i (1+\varphi_{i\bar{i}})}{\prod_i (1+\varphi_{i\bar{i}})}. \tag{2.36}$$

We can see that the left-hand side is to take arbitrarily $m-1$ terms of $\sum_i \frac{1}{1+\varphi_{i\bar{i}}}$ and multiply them together. This allows some repetition, such as the term looks like $\left(\frac{1}{1+\varphi_{1\bar{1}}}\right)^{(m-1)}$. However, on the right-hand side, there are only terms with different indices, i.e., each term is a product of $m-1$ terms that look like $\frac{1}{1+\varphi_{i\bar{i}}}$.

Thus, the right-hand side is included in the left-hand side. And according to the assumption, $(g'_{i\bar{j}})$ is positive definite, i.e. for any $i, 1+\varphi_{i\bar{i}} > 0$. Therefore, the inequality 2.35 holds.

We continue to simplify the right-hand side of 2.35.

The numerator is

$$\sum_i (1+\varphi_{i\bar{i}}) = m + \Delta\varphi. \tag{2.37}$$

Combine with the fact that φ satisfies the complex Monge-Ampère equation 2.1, in our coordinate, the left-hand side of the equation is a diagonal matrix, thus:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) = \prod_i (1+\varphi_{i\bar{i}}) = \exp\{F\}. \tag{2.38}$$

Substitute the above equation and 2.37 into 2.35, we have:

$$\sum_i \frac{1}{1+\varphi_{i\bar{i}}} \geq (m+\Delta\varphi)^{1/(m-1)} \exp\left\{\frac{-F}{m-1}\right\}. \tag{2.39}$$

Substitute the above equation into 2.34 and sort out, we find:

$$\begin{aligned}\Delta'(\exp\{-C\varphi\}(m+\Delta\varphi)) \geq{} & \exp\{-C\varphi\}\left(\Delta F - m^2 \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right) \\ & - C\exp\{-C\varphi\}m(m+\Delta\varphi) \\ & + \left(C + \inf_{i\neq l}(R_{i\bar{i}il})\right)\exp\{-C\varphi\}\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{1+1/(m-1)}.\end{aligned} \tag{2.40}$$

We can choose the constant C large enough, s.t.:

$$C + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) > 1. \tag{2.41}$$

This assumption assures the coefficient of the highest order terms w.r.t. to $m+\Delta\varphi$ in 2.40 is positive.

We know that $\exp\{-C\varphi\}(m+\Delta\varphi)$ must achieve its maximal on the compact manifold at some point p. At this point, the Laplacian is non-positive. Thus, at p, the left-hand side of the inequality 2.40 is non-positive, and we have:

$$\begin{aligned}0 \geq{} & \Delta F - m^2 \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) - Cm(m+\Delta\varphi) \\ & + \left(C + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right)\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{1+1/(m-1)}.\end{aligned} \tag{2.42}$$

Set the crucial term $m+\Delta\varphi$ to be y, then the above inequality can be regarded as an inequality w.r.t. y:

$$y^{1+1/(m-1)} \leq ay+b. \tag{2.43}$$

Thus, we can deduce that either $y^{1+1/(m-1)} \leq 2ay$ or $y^{1+1/(m-1)} \leq 2b$ is true. No matter which inequality holds, we get an upper bound estimate of y w.r.t. a, b.

Therefore, we can implement a similar method to inequality 2.42. This gives us the upper estimate of $(m+\Delta\varphi)(p)$ in terms of $\sup_M(-\Delta F)$, $\sup_M |\inf_{i\neq l}(R_{i\bar{i}l\bar{l}})|$, $C\cdot m$ and $\sup_M F$, which we denote by C_1, i.e.:

$$(m+\Delta\varphi)(p) \leq C_1, \tag{2.44}$$

where the constant C_1 only depends on the manifold M and its initial Kähler metric.

In particular, we should notice the power of the right-hand side in 2.39, which is slightly greater than $\frac{1}{m}$. In the inequality 2.35, we may wonder if we can use the mean value inequality directly and get:

$$\sum_i \frac{1}{1+\varphi_{i\bar{i}}} \geq m \sqrt[m]{\prod_i (1+\varphi_{i\bar{i}})} = m\exp\{\frac{F}{m}\}. \tag{2.45}$$

If we apply the above inequality to inequality 2.34, by similar computation, we have the following inequality that looks like 2.42:

$$\begin{aligned}
0 &\geq \Delta F - m^2 \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) - Cm(m+\Delta\varphi) + \left(C + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right) m(m+\Delta\varphi)\exp\{\frac{F}{m}\} \\
&= \Delta F - m^2 \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) + \left[\left(C + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right)\exp\{\frac{F}{m}\} - C\right] m(m+\Delta\varphi) \\
&= \Delta F - m^2 \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) + \left[\left(\exp\{\frac{F}{m}\} - 1\right) C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}} \cdot \exp\{\frac{F}{m}\}\right] m(m+\Delta\varphi).
\end{aligned} \tag{2.46}$$

In this way, the coefficient of C in the above inequality may not always be positive. However, we need C to be positive. If $\exp\{\frac{F}{m}\}-1$ is negative, as C becomes large, we cannot get an upper bound estimate of $m+\Delta\varphi$. Therefore, the inequality 2.35 is necessary and ingenious.

So far, we have the maximal estimate of $m+\Delta\varphi$ at the point p, where $\exp\{-C\varphi\}(m+\Delta\varphi)$ achieves its maximal on the whole manifold. We still want to know the estimate of $m+\Delta\varphi$ not only at p but on the whole manifold M. This relies on the uniform bound of φ:

$$\begin{aligned}
\exp\{-C\varphi\}(m+\Delta\varphi) &\leq \exp\{-C\varphi(p)\}(m+\Delta\varphi)(p) \\
&= \exp\{-C\varphi(p)\}\cdot C_1 \\
&\leq C_1 \cdot \sup_M \exp\{-C\varphi\} \\
&= C_1 \cdot \exp\{-C\inf_M \varphi\}.
\end{aligned} \tag{2.47}$$

Therefore, after shifting terms in the above inequality, and together with the positiveness of Hermitian metric, we have the second-order estimate of φ in terms of the lower bound of φ itself:

$$0 < m + \Delta\varphi \leq C_1 \exp\left\{C\left(\varphi - \inf_M \varphi\right)\right\}. \tag{2.48}$$

2.2 A Priori C^0 Estimates

In this section, we will first use the properties of Green functions to give the integral estimate of φ, i.e., the L^1-norm estimate. If we both have the L^1 and C^2 estimates of φ, with Schauder theory in elliptic equations, we will have the C^1 estimate of φ. Therefore, the C^1, C^2 estimates are reduced to the C^0 estimate. And giving the C^0 estimate is also the most crucial step in Shing-Tung Yau's paper.

First, using the Green function, we can give the upper bound estimate and L^1 estimate of φ without many difficulties.

Suppose $G(p,q)$ is the Green function of Laplacian on the compact manifold M. The classic differential equation theory assures the existence and uniqueness of the Green function. One can reference the book [Li12], Chapter 17.

And we can always find a constant K only depends on the manifold M, s.t.:

$$G(p,q) + K \geq 0. \tag{2.49}$$

Since the manifold M is compact, we don't need to consider the boundary terms. Together with the regularity assumption of φ, using the Green formula, we have, for any $p \in M$:

$$\begin{aligned} \varphi(p) &= -\int_M G(p,q)\Delta\varphi(q)dq \\ &= -\int_M (G(p,q)+K)\Delta\varphi(q)dq. \end{aligned} \tag{2.50}$$

By integral by parts, we know the integral of the Laplacian of a function is always equal to 0 on compact manifolds. Thus, the last equality holds. We add the constant K to ensure the positive sign of $G(p,q) + K$.

Recall that we have $m + \Delta\varphi > 0$, i.e. $-\Delta\varphi < m$, and substitute it into the equation above:

$$\sup_M \varphi \leq m \sup_{p\in M} \int_M (G(p,q)+K)dq. \tag{2.51}$$

Therefore, we give the upper bound estimate of φ, which only relies on the estimates of the fixed Green function. To get the estimate of $\sup_M |\varphi|$, we still need the lower bound estimate of φ, i.e., we need to give the estimate of $\inf_M \varphi$.

By the above inequality and our assumption, $\int_M \varphi = 0$, together with the triangle inequality, we find:

$$\begin{aligned}\int_M |\varphi| &= \int_M |\sup_M \varphi - \varphi - \sup_M \varphi| \leq \int_M \left|\sup_M \varphi - \varphi\right| + \int_M |\sup \varphi| \\ &= \left(\sup_M \varphi\right) \mathrm{Vol}(M) - \int_M \varphi + \left(\sup_M \varphi\right) \mathrm{Vol}(M) \\ &\leq 2m \,\mathrm{Vol}(M) \sup_{p\in M} \int_M (G(p,q)+K)dq.\end{aligned} \tag{2.52}$$

Since $\sup_M \varphi \geq \varphi$, i.e., $\sup_M \varphi - \varphi \geq 0$, we see clearly that the second line in the above computation holds. By assumption $\int_M \varphi = 0$, thus, there is a point x, s.t. $\sup_M \varphi \geq \varphi(x) \geq 0$. Therefore, we can remove the absolute value sign for $\sup_M \varphi$ in the first line.

To estimate $\sup_M |\varphi|$, we still need the lower bound estimate, i.e. $\inf_M \varphi$. In Yau's paper, he gave two different methods to prove such an estimate exists, according to whether the manifold's dimension equals 2. Firstly, he gave the estimate when $m = 2$, then the proof for arbitrary dimension.

Let's first assume $m = 2$. We have had the estimate of $\sup_M \varphi$. Thus, we can rescale to fix the sign of φ. Because we want the lower bound estimate, w.l.o.g. we assume φ is always negative. To achieve this, we can choose $\tilde{\varphi}$ differs only a constant from φ:

$$\tilde{\varphi} = \varphi - 1 - m \sup_{p\in M} \int_M (G(p,q)+K)dq. \tag{2.53}$$

Substitute φ with $\tilde{\varphi}$, and for convenience, we still write φ.

By the triangle inequality and the compactness of the manifold M, we still have the L^1 estimate, i.e., $\int_M |\varphi|$ can be estimated, and also we know $\inf_M \varphi \leq \sup_M \varphi \leq -1$.

The basic idea for the C^0 estimate is:

Firstly, consider the upper bound estimate of the integral $\int_M \exp\{k\varphi^2\}$, where k is a positive number that satisfies some constraints. Suppose we have such an upper estimate. We want a lower bound for this integral related to $\inf_M \varphi$. Then, we can estimate $\inf_M \varphi$ using the inequality. More specifically, we rescale the integral in some geodesic ball, s.t. the radius of the geodesic ball relates to $\inf_M \varphi$, and by assumption, $\exp\{k\varphi^2\}$ is lower bounded in this geodesic ball. This will turn out to give us the wanted inequality.

In order to show the upper bound estimate of $\int_M \exp\{k\varphi^2\}$, we consider the Taylor expansion of $\exp\{k\varphi^2\}$ w.r.t. $k\varphi^2$:

$$\exp\{k\varphi^2\} = \sum_{p=0}^{\infty} \frac{k^p}{p!} \varphi^{2p}. \tag{2.54}$$

Since the convergence radius of the series above is ∞, i.e. the Taylor expansion converges uniformly to $\exp\{k\varphi^2\}$, we can take the term by term integral:

$$\int_M \exp\{k\varphi^2\} \leq \sum_{p=0}^{\infty} \frac{k^p}{p!} \int_M |\varphi|^{2p}. \tag{2.55}$$

Therefore, we need to give upper bound estimates of $\int_M |\varphi|^{2p}$ for any positive integer p. For the special case, when $p = 0$, apparently we have $\int_M 1 = \mathrm{Vol}(M)$.

Firstly, by definition, suppose $f \in C^2(\mathbb{R}), g \in C^2(M; \mathbb{R})$, we have the following Laplacian formula for composite functions:

$$\Delta f(g) = f' \Delta g + f'' |\nabla g|^2. \tag{2.56}$$

Thus, for any fixed $p \geq 1$ (note that we don't require p to be an integer here), substitute $f(x) = x^p, g = -\varphi$ into the above formula. Note that we use the normalized Laplacian and inner product of the metric g' here:

$$\begin{aligned}
\Delta'(-\varphi)^p &= -p(-\varphi)^{p-1}\Delta'\varphi + p(p-1)(-\varphi)^{p-2} {g'}^{i\bar{j}} \varphi_i \varphi_{\bar{j}} \\
&= -p(-\varphi)^{p-1}\Delta'\varphi + p(p-1)(-\varphi)^{p-2} \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} \\
&= -p(-\varphi)^{p-1} \left(m - \sum_i \frac{1}{1+\varphi_{i\bar{i}}} \right) + p(p-1)(-\varphi)^{p-2} \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}.
\end{aligned} \tag{2.57}$$

By assumption, now $m = 2$. Expand the summation, we have:

$$\begin{aligned}
\Delta'(-\varphi)^p =& p(p-1)(-\varphi)^{p-2} \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} \\
& - 2p(-\varphi)^{p-1} + p(-\varphi)^{p-1} \left(1+\varphi_{1\bar{1}}\right)^{-1} \left(1+\varphi_{2\bar{2}}\right)^{-1} (2+\Delta\varphi) \\
=& p(p-1)(-\varphi)^{p-2} \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} - 2p(-\varphi)^{p-1} \\
& + 2p(-\varphi)^{p-1} \exp\{-F\} \\
& + \exp\{-F\} \left[-\Delta(-\varphi)^p + p(p-1)(-\varphi)^{p-2} |\nabla\varphi|^2 \right].
\end{aligned} \tag{2.58}$$

In the second line of the above equation, we use:

$$\begin{aligned}
\left(1+\varphi_{1\bar{1}}\right)^{-1} \left(1+\varphi_{2\bar{2}}\right)^{-1} (2+\Delta\varphi) &= \frac{\left(1+\varphi_{1\bar{1}}\right) + \left(1+\varphi_{2\bar{2}}\right)}{\left(1+\varphi_{1\bar{1}}\right)\left(1+\varphi_{2\bar{2}}\right)} \\
&= \frac{1}{1+\varphi_{1\bar{1}}} + \frac{1}{1+\varphi_{2\bar{2}}}.
\end{aligned} \tag{2.59}$$

In the last line of 2.58, we use the equation $\exp\{F\} = \left(1+\varphi_{1\bar{1}}\right)\left(1+\varphi_{2\bar{2}}\right)$ and the Laplacian formula for composite functions.

Then, we multiply 2.58 by $\exp\{F\}$ on both sides, and take integral w.r.t. the volume form of the initial metric $g = \sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. By the complex Monge-Ampère equation 2.1 and the fact that the integral of the Laplacian of a function equals 0, we have:

$$\int_M \exp\{F\}\Delta'(-\varphi)^p \omega_g^m = \int_M \Delta'(-\varphi)^p \omega_{g'}^m = 0. \tag{2.60}$$

Here, we use ω_g to denote the volume form in terms of the metric g. In this way, the equation 2.1 is equivalent to:

$$\omega_{g'}^m = \exp\{F\}\omega_g^m. \tag{2.61}$$

Shift terms, we then have:

$$\begin{aligned} 2p\int_M (-\varphi)^{p-1}(1-\exp\{F\}) =& p(p-1)\int_M (-\varphi)^{p-2}\left(\sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}\right)\exp\{F\} \\ & -\int_M \Delta(-\varphi)^p + p(p-1)\int_M (-\varphi)^{p-2}|\nabla\varphi|^2. \end{aligned} \tag{2.62}$$

The second term on the right-hand side of the above equation equals 0, and we have:

$$\begin{aligned} |\nabla(-\varphi)^{p/2}|^2 &= g(\nabla(-\varphi)^{p/2}, \nabla(-\varphi)^{p/2}) = \sum_i \frac{\partial}{\partial z^i}(-\varphi)^{p/2} \cdot \frac{\partial}{\partial \bar{z}^i}(-\varphi)^{p/2} \\ &= \frac{p^2}{4}(-\varphi)^{p-2}|\nabla\varphi|^2. \end{aligned} \tag{2.63}$$

Therefore,

$$\begin{aligned} 2p\int_M (-\varphi)^{p-1}(1-\exp\{F\}) = & \; p(p-1)\int_M (-\varphi)^{p-2}\left(\sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}\right)\exp\{F\} \\ & + \frac{4(p-1)}{p}\int_M \left|\nabla(-\varphi)^{p/2}\right|^2. \end{aligned} \tag{2.64}$$

Recall that we set φ to be less or equal to -1. Thus, we can scale the first term of the right-hand side in the above equation. Because it's greater than 0, and we get:

$$\int_M \left|\nabla(-\varphi)^{p/2}\right|^2 \le \frac{p^2}{2(p-1)}\int_M (-\varphi)^{p-1}(1-\exp\{F\}). \tag{2.65}$$

Set $C_2 = \sup_M(1-\exp\{F\})$. When $p \ge 2$, we have:

$$\frac{p}{2(p-1)} \le 1. \tag{2.66}$$

Therefore, by 2.65, we have the following useful inequality:

$$\begin{aligned} \int_M \left|\nabla(-\varphi)^{p/2}\right|^2 &\le \frac{p^2}{2(p-1)}\left|\int_M (1-\exp\{F\})(-\varphi)^{p-1}\right| \\ &\le pC_2\int_M |\varphi|^{p-1}. \end{aligned} \tag{2.67}$$

Remember that we aim to give the upper bound estimate of $\int_M |\varphi|^{2p}$ for any positive integer p.

Next, we need to use the Sobolev inequality. One can refer to [Eva10]: Suppose $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open subset in the Euclidean space. When $n > p$, set $1/p^* = 1/p - 1/n$, we have:

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega, n, p)\|\nabla u\|_{W^{1,r}(\Omega)}. \tag{2.68}$$

Since the manifold M is compact, we can always divide it into finite open subsets, and each subset is diffeomorphism to some bounded open subset in the Euclidean space. We have the Sobolev inequality in each such subset. Thus, together, we see the Sobolev inequality holds on the whole manifold M, i.e., for any function $f \in W^{1,r}, r < \dim_{\mathbb{R}}(M)$, we have:

$$\|f\|_{L^{r^*}(M)} \leq C(\|f\|_{L^r(M)} + \|\nabla f\|_{L^r(M)}), \tag{2.69}$$

where the constant C only depends on the manifold M.

Now, the complex dimension of the manifold $m = 2$ and the real dimension $n = 4$. Take $r = 2, r^* = 4$, and $f = |\varphi|^{\frac{p}{2}}$:

$$\left(\int_M |\varphi|^{2p}\right)^{1/4} \leq C\left(\left(\int_M |\varphi|^p\right)^{1/2} + \left(\int_M |\nabla(-\varphi)^{p/2}|^2\right)^{1/2}\right). \tag{2.70}$$

By assumption, φ is always negative on the manifold M. Take a square on both sides of the above inequality and use the mean value inequality to the cross-product term. Finally, substitute 2.67 into the inequality, we will get a constant $C_3 = 2C^2$ that only depends on the manifold M, s.t.:

$$\begin{aligned}
\left(\int_M |\varphi|^{2p}\right)^{1/2} &\leq C^2\left(\int_M |\varphi|^p + \int_M |\nabla(-\varphi)^{p/2}|^2 + 2\left(\int_M |\varphi|^p\right)^{1/2} \cdot \left(\int_M |\nabla(-\varphi)^{p/2}|^2\right)^{1/2}\right) \\
&\leq 2C^2\left(\int_M |\varphi|^p + \int_M \left|\nabla(-\varphi)^{p/2}\right|^2\right) \\
&= C_3\int_M |\varphi|^p + C_3\int_M \left|\nabla(-\varphi)^{p/2}\right|^2 \\
&\leq C_3\int_M |\varphi|^p + pC_2C_3\int_M |\varphi|^{p-1}.
\end{aligned} \tag{2.71}$$

Since $\varphi \leq -1, |\varphi| \geq 1$, we know $|\varphi|^{p-1} \leq |\varphi|^p$. Thus, we can scale the second term in the last line of the above inequality:

$$\int_M |\varphi|^{2p} \leq \left(C_3\int_M |\varphi|^p + pC_2C_3\int_M |\varphi|^{p-1}\right)^2 \leq (C_3 + pC_2C_3)^2\left(\int_M |\varphi|^p\right)^2. \tag{2.72}$$

In this way, we can find a constant C_4 that only depends on the manifold M, s.t.:

$$\int_M |\varphi|^{2p} \leq C_4 p^2\left(\int_M |\varphi|^p\right)^2. \tag{2.73}$$

This inequality is useful in iteration since it estimates the higher-order integral norm with only the lower-order integral norm.

Next, we need the Poincaré inequality. The reference is also [Eva10]:

Set $u \in W^{1,r}(M), \bar{u} = \frac{1}{\mathrm{Vol}(M)} \int_M u$, which is the average of the function u on the manifold. Poincaré inequality states that:

$$\int_M (u - \bar{u})^r \leq C_P \int_M |\nabla u|^r, \tag{2.74}$$

where C_P is called the Poincaré constant that only depends on the manifold M.

By assumption $\int_M \varphi = 0$. Thus, when we take $r = 2$, this inequality becomes $\int_M |\varphi|^2 \leq C \int_M |\nabla \varphi|^2$, where the constant C only depends on the manifold M. Also, together with 2.67, we have:

$$\int_M |\varphi|^2 = \int_M (-\varphi)^2 \leq C \int_M |\nabla(-\varphi)|^2 \leq 2CC_2 \int_M |\varphi|. \tag{2.75}$$

Therefore, we can get the L^2 estimate of φ with the previous L^1 estimate.

Next, we want to show that for any integer $p \geq 1$, there is a constant C_5 that only depends on the manifold M, s.t.:

$$\int_M |\varphi|^p \leq C_5^p \left(\frac{p}{2}\right)^{p/2}. \tag{2.76}$$

Set p_0 is the first integer, satisfies that for any $p \geq p_0$, we have:

$$C_4(1 + \mathrm{Vol}\, M) \left(\frac{p+2}{4}\right)^{(p+1)/2} \leq \left(\frac{p+1}{2}\right)^{(p-3)/2}. \tag{2.77}$$

Such a p_0 always exists since we can shift the terms to the right-hand side and see:

$$\lim_{p\to\infty} \frac{\left(\frac{p+1}{2}\right)^{(p-3)/2}}{\left(\frac{p+2}{4}\right)^{(p+1)/2}} = \lim_{p\to\infty} e^{-1/2}(1 - \frac{1}{p+2})^{-5/2} \cdot 2^{\frac{p+5}{2}} \cdot \frac{1}{(p+2)^2} = +\infty. \tag{2.78}$$

The first two terms in the above limit go to a constant $e^{-1/2}$, and the last two terms go to $+\infty$. Thus, we know the limit is $+\infty$.

On the other hand, we already had the estimates of $\|\varphi\|_{L^1}, \|\varphi\|_{L^2}$. In order to get the higher estimates of $\|\varphi\|_{L^p}$, we only need to use the Hölder inequality repeatedly. Suppose $\Omega \subset \mathbb{R}^n, f \in L^p(\Omega), g \in L^q(\Omega), \frac{1}{p} + \frac{1}{q} = 1, 1 \leq p, q \leq \infty$, we have:

$$\int_\Omega |fg| dx \leq \|f\|^p \cdot \|f\|^p. \tag{2.79}$$

With above inequality and the useful inequality 2.73, we have the $\|\varphi\|_{L^p}$ estimates w.r.t. arbitrary positive integer p.

For example, we can consider a concrete norm:

The estimate of $\|\varphi\|_{L^5}$. By Hölder inequality:

$$\int_M |\varphi|^5 = \int_M |\varphi|^2 \cdot |\varphi|^3 \leq \left(\int_M |\varphi|^4\right)^{1/2} \cdot \left(\int_M |\varphi|^6\right)^{1/2}. \tag{2.80}$$

Again, use the inequality 2.73 and the Hölder inequality, we will get an estimate of $\|\varphi\|_{L^5}$ that only depends on the estimates of $\|\varphi\|_{L^1}, \|\varphi\|_{L^2}$.

We still need to know how fast the estimate of $\|\varphi\|_{L^p}$ grows w.r.t. p.

With the above example in mind, we can describe the idea first:

We want to use the lower integral norm of φ to control the higher integral norm. The most powerful tool is the inequality 2.73.

We continue the argument according to the different parity of $p+1$, and we can get the result quickly:

First, take a sufficiently large constant C_5, s.t. for any p satisfies $1 \leq p \leq 2p_0$, for all these finitely many terms, the inequality 2.76 holds. Then, by mathematical induction, we want to prove that 2.76 holds for any $p \geq 1$.

If $p+1$ is even, we first apply the inequality 2.73. Then, by the induction assumption, we apply the inequality 2.76 to get:

$$\begin{aligned} \int_M |\varphi|^{p+1} &\leq C_4 \left(\frac{p+1}{2}\right)^2 \left(\int_M |\varphi|^{(p+1)/2}\right)^2 \\ &\leq C_4 \left(\frac{p+1}{2}\right)^2 \left(C_5^{(p+1)/2} \left(\frac{p+1}{4}\right)^{(p+1)/4}\right)^2 \\ &= C_4 \left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+1}{4}\right)^{(p+1)/2}. \end{aligned} \tag{2.81}$$

Use 2.77 to scale the above inequality:

$$\begin{aligned} \int_M |\varphi|^{p+1} &\leq C_4 \left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+1}{4}\right)^{(p+1)/2} \\ &\leq \left(\frac{p+1}{2}\right)^2 \cdot C_4 \left(\frac{p+2}{4}\right)^{(p+1)/2} \cdot C_5^{p+1} \\ &\leq \left(\frac{p+1}{2}\right)^2 \cdot \frac{1}{(1+\mathrm{Vol}(M))} \left(\frac{p+1}{2}\right)^{(p-3)/2} \cdot C_5^{p+1} \\ &\leq C_5^{p+1} \left(\frac{p+1}{2}\right)^{(p+1)/2}. \end{aligned} \tag{2.82}$$

Thus, we know if we use $p+1$ to replace p, the inequality 2.76 still holds.

If $p+1$ is odd, we also first use the inequality 2.73, which allows us to descend the order of the integral norm of φ. Then, we use the Hölder inequality to scale

$|\varphi|^{(p+1)/2}$ to $|\varphi|^{(p+2)/2}$. Thus, we can still use the induction assumption in 2.76. Here is the computation:

$$
\begin{aligned}
\int_M |\varphi|^{p+1} &\leq C_4 \left(\frac{p+1}{2}\right)^2 \left(\int_M |\varphi|^{(p+1)/2}\right)^2 \\
&\leq C_4 \left(\frac{p+1}{2}\right)^2 \left(\int_M |\varphi|^{(p+1)/2} \cdot 1\right)^2 \\
&\leq C_4 \left(\frac{p+1}{2}\right)^2 \left(\left(\int_M \left(|\varphi|^{\frac{p+1}{2}}\right)^{\frac{p+2}{p+1}}\right)^{\frac{p+1}{p+2}} \cdot \left(\int_M 1^{p+2}\right)^{\frac{1}{p+2}}\right)^2 \\
&= C_4 \left(\frac{p+1}{2}\right)^2 \left[\int_M |\varphi|^{(p+2)/2}\right]^{2(p+1)/(p+2)} \mathrm{Vol}(M)^{2/(p+2)} \\
&\leq C_4 \left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+2}{4}\right)^{(p+1)/2} \mathrm{Vol}(M)^{2/(p+2)}.
\end{aligned}
\tag{2.83}
$$

Note that in the third line of the above inequality, we take the power in Hölder inequality to be $\frac{p+2}{p+1}, p+2$, respectively, s.t.:

$$
\frac{p+1}{p+2} + \frac{1}{p+2} = 1. \tag{2.84}
$$

Thus, we know that when $p+1$ is odd, the inequality 2.76 holds.

Therefore, there is a large constant C_5, s.t. for any positive integer $p \geq 1$, the inequality 2.76 holds.

According to the idea stated above, we can apply the inequality 2.76 to 2.55:

$$
\begin{aligned}
\int_M \exp\left\{k\varphi^2\right\} &\leq \sum_{p=0}^{\infty} \frac{k^p}{p!} \int_M |\varphi|^{2p} \\
&\leq \sum_{p=0}^{\infty} \frac{k^p}{p!} \left(C_5^2\right)^p p^p.
\end{aligned}
\tag{2.85}
$$

By the Stirling's formula, for any positive integer p:

$$
p! \geq \sqrt{2\pi} p^{p+1/2} e^{-p}. \tag{2.86}
$$

Plug the above inequality into 2.85, we have:

$$
\begin{aligned}
\int_M \exp\left\{k\varphi^2\right\} &\leq \sum_{p=0}^{\infty} \frac{k^p}{\sqrt{2\pi} p^{p+1/2} e^{-p}} \left(C_5^2\right)^p p^p \\
&= \sum_{p=0}^{\infty} (2\pi)^{-1/2} \left(kC_5^2 e\right)^p p^{-1/2}.
\end{aligned}
\tag{2.87}
$$

Set $a := kC_5^2 e$. We consider the convergence of the following series:

$$
\sum_{p=0}^{\infty} \frac{a^p}{p^{1/2}}. \tag{2.88}
$$

By the ratio criterion:

$$\lim_{p\to\infty}\left|\frac{a^{p+1}}{(p+1)^{1/2}}/\frac{a^p}{p^{1/2}}\right| = a. \tag{2.89}$$

Thus, if we want the series to converge, we need to restrict k, s.t. $k < C_5^{-2}e^{-1}$, i.e. $a < 1$. In this case, we know the right-hand side series of 2.87 is convergent. Therefore, we have an upper estimate of $\int_M \exp\{k\varphi^2\}$.

Next, we can give the estimate of $\sup_M |\varphi|$: Set $\Delta\varphi = f$, then by the inequality 2.48:

$$-m \le f \le C_1 \exp\{C\sup\varphi\}\exp\left\{-\inf_M \varphi\right\}, \tag{2.90}$$

where C_1, C are known positive constant.

By Schauder estimates, one can refer to [Mor66] p.156(5.5.23), we know there is a constant C_6 that only depends on the manifold M and $C_1\exp\{C\sup\varphi\}$, s.t.:

$$\sup_M |\nabla\varphi| \le C_6\left(\exp\{-C\inf\varphi\} + \int_M |\varphi|\right). \tag{2.91}$$

Together with the previous estimates 2.51 2.52, and the estimates for $\|\varphi\|_{L^1}$ and $\sup_M \varphi$, we can scale the right-hand side of the above inequality. Thus, we get a constant C_7 that only depends on the manifold M and C, s.t.:

$$\sup_M |\nabla\varphi| \le C_7(\exp\{-C\inf\varphi\} + 1). \tag{2.92}$$

As stated before, we consider a special geodesic ball on the manifold M:

Set q is to be a minimal point of φ in the manifold M, i.e. $\varphi(q) = \inf_M \varphi$. Consider the geodesic ball, centered at q, and with radius $r = -\frac{1}{2}(\inf\varphi)C_7^{-1}(\exp\{-C\inf\varphi\}+1)^{-1}$. Note that $-\inf_M\varphi$ is assumed to be a large positive number. Otherwise there is an estimate for $\inf_M\varphi$ anyway. Thus, we can further assume the radius of this geodesic ball is smaller than the injective radius of the point q. Then, by the mean value theorem, since we have an estimate for $\nabla\varphi$, in this geodesic ball, we always have $\varphi \le \frac{1}{2}\inf_M\varphi$.

We scale smaller the upper bound estimate of $\int_M \exp\{k\varphi^2\}$, at any point on the manifold M, we have:

$$\exp\left\{k\varphi^2\right\} \ge \exp\left\{k(\frac{1}{2}\inf_M\varphi)^2\right\}. \tag{2.93}$$

Therefore,

$$\int_{B(q,r)}\exp\left\{k\varphi^2\right\} \ge C_8\exp\left\{\frac{1}{4}k(-\inf\varphi)^2\right\}\left(-\frac{1}{2}\inf\varphi\right)^{2m}C_7^{-2m}(\exp\{-C\inf\varphi\}+1)^{-2m}, \tag{2.94}$$

where C_8 is a positive constant that only depends on the manifold M.

Because the right-hand side can be regarded as an increasing function of $-\inf_M\varphi$, and goes to $+\infty$, as $-\inf_M\varphi$ goes to $+\infty$, and now we know the right-hand side has an upper bound. As a result, we know $-\inf_M \varphi$ is upper bounded, and $\inf_M \varphi$ is lower bounded. Together with the estimate for $\sup_M \varphi$, we get the estimate for $\sup_M |\varphi|$, i.e. the C^0 estimate, when $m = 2$.

Next, we don't assume that $m = 2$, and give the estimate of $\inf_M \varphi$ for general dimension.

Suppose N is some positive integer. In order to be different with the inequality 2.34, we replace the C there with N, and get:

$$\begin{aligned}\Delta'(\exp\{-N\varphi\}(m+\Delta\varphi)) \geq \exp\{-N\varphi\}&\left(\Delta F - m^2 \inf_{i\neq l}\left(R_{i\bar{i}l\bar{l}}\right)\right)\\ &- N\exp\{-N\varphi\}m(m+\Delta\varphi)\\ &+ \left(N + \inf_{i\neq l}\left(R_{i\bar{i}l\bar{l}}\right)\right)\exp\{-N\varphi\}(m+\Delta\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right).\end{aligned} \tag{2.95}$$

Similar to the previous argument, to fix the sign and to compute further estimates, we need to set N large enough, s.t.:

$$N + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) \geq \frac{1}{2}N. \tag{2.96}$$

Recall we have the conclusion 2.39, which is obtained by using the special inequality 2.35. Multiply the following term on both sides of 2.39:

$$\left(N + \inf_{i\neq l}\left(R_{i\bar{i}l\bar{l}}\right)\right)(m+\Delta\varphi),$$

and we have:

$$\left(N + \inf_{i\neq l}\left(R_{i\bar{i}l\bar{l}}\right)\right)(m+\Delta\varphi)\left(\sum_i \frac{1}{1+\varphi_{i\bar{i}}}\right) \geq \frac{1}{2}N\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{1+\frac{1}{m-1}}. \tag{2.97}$$

Also, we use the fact that if $x \in \mathbb{R}$ is bounded from below, there is some constant c, s.t.:

$$x^{1+\frac{1}{m-1}} - x \geq c. \tag{2.98}$$

With the inequality above, and together with the positiveness of the Kähler metric, set x to be $m + \Delta\varphi$. We know there is a constant C_9 that only depends on $\sup F$ and m, s.t.:

$$\frac{1}{2}N\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{m/(m-1)} \geq 2Nm(m+\Delta\varphi) - NC_9. \tag{2.99}$$

Substitute 2.96, 2.97 and 2.119 into 2.95, we have:

$$\begin{aligned}\Delta'(\exp\{-N\varphi\}(m+\Delta\varphi)) \geq{}& \exp\{-N\varphi\}\left(\Delta F - m^2 \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right) - N\exp\{-N\varphi\}m(m+\Delta\varphi)\\ &+ \frac{1}{2}N\exp\left\{\frac{-F}{m-1}\right\}(m+\Delta\varphi)^{\frac{m}{m-1}}\exp\{-N\varphi\}\\ \geq{}& \exp\{-N\varphi\}\left(\Delta F - m^2\inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) - NC_9\right) + N\exp\{-N\varphi\}m(m+\Delta\varphi).\end{aligned} \tag{2.100}$$

Multiply $\exp\{F\}$ on both sides, scale a little bit, and multiply the coefficient into the bracket of the last term $m+\Delta\varphi$. We have:

$$\begin{aligned}\exp\{F\}\Delta'(\exp\{-N\varphi\}(m+\Delta\varphi)) \geq{}& \exp\{-N\varphi\}\exp\{F\}\left(\Delta F - m^2\inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) - NC_9\right)\\ &+ N\exp\{-N\varphi\}\exp\{\inf F\}m(m+\Delta\varphi)\\ ={}& \exp\{-N\varphi\}\left\{\exp\{F\}\left(\Delta F - m^2\inf_{i\neq l}(R_{i\bar{i}i\bar{l}}) - NC_9\right) + m^2N\exp\{\inf F\}\right\}\\ &+ mN\exp\{\inf F\}\exp\{-N\varphi\}\Delta\varphi.\end{aligned} \tag{2.101}$$

Then, compute the Laplacian of the composite function $\exp\{-N\varphi\}$:

$$\Delta\exp\{-N\varphi\} = -N\exp\{-N\varphi\}\Delta\varphi + N^2\exp\{-N\varphi\}|\nabla\varphi|^2. \tag{2.102}$$

Plug the first term on the right-hand side of the above equality into the last term on the right-hand side of 2.101:

$$\begin{aligned}&\exp\{F\}\Delta'(\exp\{-N\varphi\}(m+\Delta\varphi))\\ &\quad\geq \exp\{-N\varphi\}\left\{\exp\{F\}\left(\Delta F - m^2\inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) - NC_9\right) + m^2N\exp\{\inf F\}\right\}\\ &\quad+ m\exp\{\inf F\}\left(-\Delta\exp\{-N\varphi\} + N^2\exp\{-N\varphi\}|\nabla\varphi|^2\right).\end{aligned} \tag{2.103}$$

The terms in the brace of the above inequality are bounded from below on the compact manifold M. We denote it as $-C_{10}$, and it only depends on N, F and the manifold M:

$$\begin{aligned}&\exp\{F\}\Delta'(\exp\{-N\varphi\}(m+\Delta\varphi))\\ &\geq -C_{10}\exp\{-N\varphi\} + m\exp\{\inf F\}\left(-\Delta\exp\{-N\varphi\} + N^2\exp\{-N\varphi\}|\nabla\varphi|^2\right).\end{aligned} \tag{2.104}$$

Multiply the above inequality with the volume form ω_g^m w.r.t. the initial Kähler metric and take the integral. By the complex Monge-Ampère equation $\omega_{g'}^m = \exp\{F\}\omega_g^m$, we know the left-hand side and the second term on the right-hand side are both equal to 0. Furthermore, multiply $\frac{1}{4}$ on both sides, we have:

$$\begin{aligned}\frac{1}{4}N^2\int_M \exp\{-N\varphi\}|\nabla\varphi|^2 ={}& \int_M \left|\nabla\exp\left\{-\frac{1}{2}N\varphi\right\}\right|^2\\ \leq{}& \frac{1}{4}C_{10}m^{-1}\exp\{-\inf F\}\int_M \exp\{-N\varphi\}.\end{aligned} \tag{2.105}$$

With these preparations, we will give the upper bound estimate of $\int_M \exp\{-N\varphi\}$.

We use the contradiction argument. Suppose there is no upper bound for $\int_M \exp\{-N\varphi\}$, i.e. $\int_M \exp\{-N\varphi\} = +\infty$. Recall that we have some regularity assumption of φ. Thus, according to the Sobolev approximation theorem, refer to [Eva10]. We know there is a sequence of $\{\varphi_i\}$ converges to φ in the $H^1(M)$ norm. And each term satisfies the L^1 estimate in 2.52 and the above inequality. However, the limit of its integral goes to infinity:

$$\lim_{i\to\infty} \int_M \exp\{-N\varphi_i\} = +\infty. \tag{2.106}$$

We define:

$$\exp\{-N\tilde{\varphi}_i\} := \frac{\exp\{-N\varphi_i\}}{\int_M \exp\{-N\varphi_i\}}, \tag{2.107}$$

and

$$\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\} := \frac{\exp\left\{-\frac{1}{2}N\varphi_i\right\}}{\left(\int_M \exp\{-N\varphi_i\}\right)^{\frac{1}{2}}}. \tag{2.108}$$

In particular, we can compute:

$$\int_M \left|\nabla \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right|^2 = \frac{\int_M \left|\nabla \exp\left\{-\frac{1}{2}N\varphi_i\right\}\right|^2}{\int_M \exp\{-N\varphi_i\}}. \tag{2.109}$$

Therefore, by the inequality 2.105, we have:

$$\int_M \left|\nabla \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right|^2 \le \frac{1}{4}C_{10}m^{-1}\exp\{-\inf F\}. \tag{2.110}$$

From the above inequality, we know that $\int_M \left|\nabla \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right|^2$ is uniformly upper bounded, and the upper bound only depends on the constant N, F and the manifold M itself.

By the Sobolev embedding theorem [Eva10], we know that $H^1(M) \subset\subset L^2(M)$ is compact embedding, i.e. for any bounded series in $H^1(M)$, there is a subsequence that converges in the $L^2(M)$ norm, and the limit function still belongs to $L^2(M)$.

By definition 2.108, we have:

$$\int_M \left|\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right|^2 = \frac{\int_M \exp\left\{-\frac{1}{2}N\varphi_i \cdot 2\right\}}{\int_M \exp\{-N\varphi_i\}} = 1. \tag{2.111}$$

Therefore, we consider the sequence $\{\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\}$, and by the inequalities 2.110 and 2.111, we know this sequence is bounded in $H^1(M)$. Thus, a subsequence converges in the $L^2(M)$ norm. Moreover, this subsequence's limit function f still belongs to $L^2(M)$. W.l.o.g, we denote this subsequence as $\{\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\}$ itself.

On the other hand, for any $\lambda > 0$, we have:

$$\begin{aligned}\int_M |\varphi| &= \int_{\{x||\varphi(x)|\geq\lambda\}} |\varphi| + \int_{\{x||\varphi(x)|<\lambda\}} |\varphi| \\ &\geq \lambda \mathrm{Vol}\{x \mid |\varphi(x)| \geq \lambda\}.\end{aligned} \tag{2.112}$$

Rewrite the above inequality:

$$\mathrm{Vol}\{x||\varphi|(x) \geq \lambda\} \leq \frac{1}{\lambda}\int_M |\varphi|. \tag{2.113}$$

Furthermore, according to the definition 2.108, write $\exp\{-\frac{1}{2}N\tilde{\varphi}_i\}$ explicitly, and take natural logarithm on both sides, then tidy up the result:

$$\begin{aligned}\mathrm{Vol}\left\{x \,\middle|\, \lambda \leq \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right\} &= \mathrm{Vol}\left\{x \,\middle|\, \lambda \leq \frac{\exp\left\{-\frac{1}{2}N\varphi_i\right\}}{\left(\int_M \exp\left\{-N\varphi_i\right\}\right)^{\frac{1}{2}}}\right\} \\ &= \mathrm{Vol}\left\{x \,\middle|\, \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M \exp\left\{-N\varphi_i\right\} \leq -\varphi_i\right\}.\end{aligned} \tag{2.114}$$

By assumption, we have $\lim_{i\to\infty}\int_M \exp\left\{-\frac{1}{2}N\varphi_i\right\} = \infty$. Thus, when i is sufficiently large:

$$\begin{aligned}\mathrm{Vol}\left\{x \,\middle|\, \lambda \leq \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right\} &\leq \mathrm{Vol}\left\{x \,\middle|\, 0 < \frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M \exp\left\{-N\varphi_i\right\} \leq \middle| \varphi_i \middle| \right\} \\ &\leq \left(\frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M \exp\left\{-N\varphi_i\right\}\right)^{-1}\int_M |\varphi_i|.\end{aligned} \tag{2.115}$$

The first inequality holds because when i is large enough, it first assures that

$$\frac{2}{N}\log\lambda + \frac{1}{N}\log\int_M \exp\{-N\varphi_i\} > 0,$$

and if we replace $-\varphi_i$ with $|\varphi_i|$ in 2.114, the measure is non-decreasing. The second inequality holds because of the inequality 2.113.

By assumption 2.52, the sequence $\left\{\int_M |\varphi_i|\right\}$ is uniformly bounded. Thus, let i tend to ∞ in 2.115, the numerator is bounded, the denominator tends to infinity, and the whole right-hand side tends to 0. Therefore, for any $\lambda > 0$, we have:

$$\lim_{i\to\infty}\mathrm{Vol}\left\{x \mid \lambda \leq \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right\} = 0. \tag{2.116}$$

Next, we consider the measure of the set $\{x \mid \lambda \leq f(x)\}$: Firstly, since f is the $L^2(M)$ limit of the sequence $\{\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\}$, we have $f \geq 0$. We then prove the following relation:

$$\{x \mid \lambda \leq f(x)\} \subset \left\{x \mid \frac{1}{2}\lambda \leq \left|f - \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right|\right\} \bigcup \left\{x \mid \frac{1}{2}\lambda \leq \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right\}. \tag{2.117}$$

This is because if $\left|f - \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right| < \frac{1}{2}\lambda$ and $\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\} < \frac{1}{2}\lambda$, we will have:

$$f = |f| \leq \left|f - \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right| + \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\} < \lambda. \tag{2.118}$$

Thus, if $f(x) \geq \lambda$, x must belong to at least one set of the right-hand side of 2.117. Then, by the subadditivity, we have:

$$\begin{aligned}\mathrm{Vol}\{x \mid \lambda \leq f(x)\} &\leq \mathrm{Vol}\left\{x \left| \frac{1}{2}\lambda \leq \left| f - \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right|, \right.\right\} + \mathrm{Vol}\left\{x \left| \frac{1}{2}\lambda \leq \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right.\right\} \\ &\leq \frac{4}{\lambda^2}\int_M \left|f - \exp\left\{\frac{1}{2}N\tilde{\varphi}_i\right\}\right|^2 + \mathrm{Vol}\left\{x \left| \frac{1}{2}\lambda \leq \exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}\right.\right\}.\end{aligned} \tag{2.119}$$

By assumption, $\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}$ converges to a function $f \in L^2(M)$ in the L^2 norm. Thus, the first term in the second line of 2.119 tends to 0 as $i \to \infty$. And from 2.116, we know the second term in the second line of 2.119 also equals 0. Therefore, for any $\lambda > 0$, we have:

$$\mathrm{Vol}\{x|\lambda \leq f\} = 0. \tag{2.120}$$

Because f is the L^2 limit of $\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}$, $f \geq 0$. From the above equality, we know that f equals 0, a.e. on the manifold M. However, by 2.111, we know the $L^2(M)$ norm of $\exp\left\{-\frac{1}{2}N\tilde{\varphi}_i\right\}$ always equals to 1. Thus, $\int_M f^2 = 1$ which apparently contradicts with 2.120.

From the process of the computation above, we can see that it only relates to the constant N, F and the manifold M. Therefore, we not only proved the existence of the upper bound estimate of $\int_M \exp\{-N\varphi\}$ but also know the bound relies on the constant N, F and the manifold M.

Then, we use a similar argument of the case $m = 2$. The Schauder estimate holds for any dimension. Therefore, we can find a constant C_7 that only depends on the constant C and the manifold M, s.t.:

$$\sup_M |\nabla\varphi| \leq C_7(\exp\{-C\inf\varphi\} + 1). \tag{2.121}$$

We still consider the geodesic ball that centered at the minimal point q of φ, and radius $-\frac{1}{2}\inf\varphi C_7^{-1}(\exp\{-C\inf\varphi\}+1)^{-1}$. Similarly, we can suppose the radius of this ball is smaller than the injective radius of q. The mean value theorem shows that φ is no greater than $\frac{1}{2}\inf_M\varphi$ inside this ball.

When N is large enough, we have:

$$\int_{B(q,r)} \exp\{-N\varphi\} \geq C_{12}\exp\left\{-\frac{1}{2}N\inf\varphi\right\}\left(-\frac{1}{2}\inf\varphi\right)^{2m} C_7^{-2m}(\exp\{-C\inf\varphi\}+1)^{-2m}, \tag{2.122}$$

where C_{12} is a positive constant that only depends on the manifold M. Now, we proved that $\int_M \exp\{-N\varphi\}$ has an upper bound that only depends on N, F and the manifold M. Since the right-hand side of the above inequality is an increasing function w.r.t. $-\inf_M \varphi$ and tends to $+\infty$ as $-\inf_M \varphi$ tends to infinity. Now, we proved it's upper bounded. Furthermore, we know that $-\inf_M \varphi$ is upper bounded. Therefore, we have the estimate of $\sup_M |\varphi|$, i.e. the C^0 estimate of φ.

By far, we have the estimates for $\sup_M |\varphi|$, $\sup_M |\nabla\varphi|$ and $\sup_M (m+\Delta\varphi)$. Since $(g_{i\bar{j}} + \varphi_{i\bar{j}})$ is a positive definite Hermitian Matrix, in our special coordinate, for any i $1+\varphi_{i\bar{i}}$ is always positive, and we have:

$$1+\varphi_{i\bar{i}} \leq m+\Delta\varphi = \sum_i (1+\varphi_{i\bar{i}}). \tag{2.123}$$

Together with the estimate of $\sup_M (m+\Delta\varphi)$, we get the upper bound estimate for every $1+\varphi_{i\bar{i}}$.

Also, $\prod_{i=1}^m (1+\varphi_{i\bar{i}}) = \exp\{F\}$. If there is some i, s.t. $1+\varphi_{i\bar{i}}$ tends to 0. Then, there must exist some j, s.t. $1+\varphi_{j\bar{j}}$ tends to ∞ which is not possible. Therefore, a positive lower bound exists for every $1+\varphi_{i\bar{i}}$.

Thus, we proved that the metric $\sum_{i,j}(g_{i\bar{j}} + \partial^2\varphi/\partial z^i \partial \bar{z}^j) dz^i \otimes d\bar{z}^j$ is uniformly equivalent to $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$.

To sum up, we have proved the following proposition:

Proposition 2. *Let M be a compact Kähler manifold, and its Kähler metric is $g = \sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Suppose $\varphi \in C^4(M)$ is a real-valued function, s.t.:*

$$\int_M \varphi = 0, \tag{2.124}$$

and

$$\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}) dz^i \otimes d\bar{z}^j \tag{2.125}$$

defines another Kähler metric of the manifold M. If for some real-valued smooth function $F \in C^\infty(M)$, we have:

$$\det(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}) \det(g_{i\bar{j}})^{-1} = \exp F. \tag{2.126}$$

Then, there are constants C_1, C_2, C_3 and C_4 that only depend on $\inf_M F, \sup_M F, \inf_M \Delta F$ and the manifold M, s.t. $\sup_M |\varphi| \leq C_1, \sup_M |\nabla\varphi| \leq C_2, 0 < C_3 \leq 1+\varphi_{i\bar{i}} \leq C_4, \forall i$.

2.3 A Priori C^3 Estimates

We follow the method used by Calabi [Cal58], consider the function:

$$S = \sum g'^{i\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}}. \tag{2.127}$$

We consider things in the special coordinate since the above expression is independent of the local coordinate. Thus, at one point, the metric matrix $(g_{i\bar{j}})$ equals to the identity matrix, and $(\varphi_{i\bar{j}})$ is a diagonal matrix. By a rather complicated computation, we have:

$$\begin{aligned}\Delta' S \cong \sum &(1+\varphi_{i\bar{i}})^{-1}(1+\varphi_{j\bar{j}})^{-1}(1+\varphi_{k\bar{k}})^{-1}\left(1+\varphi_{\alpha\bar{\alpha}}\right)^{-1} \\ &\times \left\{\left|\varphi_{\bar{i}j\bar{k}\alpha} - \sum_p \varphi_{\bar{i}p\bar{k}}\varphi_{\bar{p}j\alpha}\left(1+\varphi_{p\bar{p}}\right)^{-1}\right|^2\right. \\ &\left. + \left|\varphi_{i\bar{j}k\alpha} - \sum_p \left(\varphi_{\bar{p}i\alpha}\varphi_{p\bar{j}k} + \varphi_{\bar{p}ik}\varphi_{p\bar{j}\alpha}\right)\left(1+\varphi_{p\bar{p}}\right)^{-1}\right|^2\right\},\end{aligned} \tag{2.128}$$

where $A \cong B$ stands for $|A-B| \leq C_1 S + C_2\sqrt{S} + C_3$, and the constants C_1, C_2, C_3 are known.

On the other hand, according to previous computation, we have:

$$\Delta'(\Delta\varphi) \geq \sum \left(1+\varphi_{k\bar{k}}\right)^{-1}\left(1+\varphi_{i\bar{i}}\right)^{-1}\left|\varphi_{k\bar{i}\bar{j}}\right|^2 - C_4, \tag{2.129}$$

where C_4 is a constant that can be estimated.

We suppose C_5 is a large enough constant, s.t.:

$$\Delta'\left(S + C_5\Delta\varphi\right) \geq C_6 S - C_7, \tag{2.130}$$

where C_6, C_7 are both constants that can be estimated.

Add $C_6C_5\Delta\varphi$ on both sides of the above inequality, and we have:

$$\Delta'\left(S + C_5\Delta\varphi\right) + C_6C_5\Delta\varphi \geq C_6S - C_7 + C_6C_5\Delta\varphi. \tag{2.131}$$

At the maximal point of $S + C_5\Delta\varphi$, we know $\Delta'\left(S + C_5\Delta\varphi\right) \leq 0$. Thus, after shifting terms, we have:

$$C_6(S + C_5\Delta\varphi) \leq C_7 + C_6C_5\Delta\varphi. \tag{2.132}$$

Since we have the uniform estimates for $\Delta\varphi$, the right-hand side of the above inequality is bounded. Thus, we have the estimate for $\sup_M(S + C_7\Delta\varphi)$. This turns out to give the estimate of $\sup_M S$. And remember that in our special coordinate, $(g_{i\bar{j}})$ and $(\varphi_{i\bar{j}})$ are both diagonal matrices, we have the estimates of $\varphi_{i\bar{j}k}$ for any i, j, k. We note that adding the term $C_6C_5\Delta\varphi$ on both sides of 2.130 helps us have the term $S + C_5\Delta\varphi$.

To sum up, we have the following proposition:

Proposition 3. *Set M is a compact Kähler manifold, with Kähler metric $g = \sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Suppose $\varphi \in C^4(M)$ is a real-valued function, s.t.:*

$$\int_M \varphi = 0, \tag{2.133}$$

and $\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j})dz^i \otimes d\bar{z}^j$ *defines another Kähler metric of the manifold* M*.*

If φ *satisfies the following equation where* $F \in C^\infty(M)$ *is a real-valued smooth function :*

$$\det(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}) \det(g_{i\bar{j}})^{-1} = \exp F. \tag{2.134}$$

Then, there is estimates for $\varphi_{i\bar{j}k}$ *that only depends on* $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$, $\sup_M |F|$, $\sup_M |\nabla F|$, $\sup_M \sup_i |F_{i\bar{i}}|$ *and* $\sup_M \sup_i |F_{i\bar{j}k}|$.

Chapter 3

Solution to the Complex Monge-Ampère Equation

Now, we can continue to prove the regularity of the a priori solution and get a smooth solution of the complex Monge-Ampère equation:

$$\det(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}) \det(g_{i\bar{j}})^{-1} = \exp F, F \in C^{\infty}(M; \mathbb{R}). \tag{3.1}$$

Take integrals on both sides of the equation. We know that the Kähler metrics in the same Kähler class give us the same volume of the manifold by taking the integral of the volume form. In fact, we can rewrite the above equation:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^m = \exp\{F\}\omega^m. \tag{3.2}$$

Expand the left-hand side, and by integral by parts, we can see why this is true.

Thus, we know the necessary condition for the function F is

$$\int_M \exp F = \mathrm{Vol}(M). \tag{3.3}$$

Next, we will prove that if $F \in C^k(M), \forall k \geq 3$ and F satisfies the condition above. Then, there is a solution φ, satisfies the equation 3.1, and $\varphi \in C^{k+1,\alpha}(M), \forall 0 \leq \alpha < 1$.

We will use the classic continuity method to prove the existence of such a solution, which is a useful tool in solving non-linear differential equations. To use this method, first, we need to consider a family of parametrized solvable complex Monge-Ampère equation, s.t. we clearly know the equation is solvable at one side of the interval of the parameter t. Then, we want to show that the set of t is both open and closed. This demonstrates that at the other side of the interval, the complex Monge-Ampère equation is still solvable.

Explicitly speaking, we consider the set:

$$S = \left\{ t \in [0,1] \mid \text{ The equation } \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \det\left(g_{i\bar{j}}\right)^{-1} \right.$$
$$\left. = \operatorname{Vol}(M) \left[\int \exp\{tF\}\right]^{-1} \exp\{tF\} \text{ has a solution } \varphi \in C^{k+1,\alpha}(M) \right\}. \tag{3.4}$$

We can directly see that $0 \in S$. Since when $t = 0$, every arbitrary constant function φ satisfies the equation.

If we can further show that the set S is both open and closed, S must be the whole set, i.e., $S = [0,1]$. And $1 \in S$ means there is a solution $\varphi \in C^{k+1,\alpha}(M)$ for the complex Monge-Ampère equation.

First, in order to show S is open, we use the standard inverse function theorem [Sch69]:

Theorem 1 (Inverse Function Theorem)**.** *Set X, Y as two Banach manifolds, $x_0 \in X, y_0 \in Y$, and let $G : X \to Y$ be a map with C^1-Frechet derivative. Suppose: (1) $G(x_0) = y_0$; (2) $D_{x_0}G : T_{x_0}X \to T_{y_0}Y$ is an isomorphism between Banach spaces. Then, there are open neighborhoods U, V, s.t. $x_0 \in U \subset X, y_0 \in V \subset Y$, and a map $F : V \to U$ with C^1-Frechet derivative and $G(F(y)) = y, \forall y \in V$.*

Although we didn't give rigorous definitions for some concepts, we can still make an analogy with the inverse function theorem in the Euclidean space when using this theorem. The application is straightforward.

We set:

$$\theta := \left\{ \varphi \in C^{k+1,\alpha}(M) \mid 1 + \varphi_{i\bar{i}} > 0, \forall i, \text{ and } \int_M \varphi = 0 \right\}, \tag{3.5}$$

and

$$B := \left\{ f \in C^{k-1,\alpha}(M) \mid \int_M f = \operatorname{Vol} M \right\}. \tag{3.6}$$

By definition, we know that θ is an open set in the Banach space $C^{k+1,\alpha}(M)$, and B is an affine hyperplane in the Banach space $C^{k-1,\alpha}(M)$. In fact, for any function $f \in B$, consider the function $f - 1$, we have $\int_M (f-1) = 0$. Thus, after the translation of B, we will get a linear hyperplane in the Banach space that passes the origin.

Consider the map $G : \theta \to B$:

$$G(\varphi) = \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1}. \tag{3.7}$$

To use the inverse function theorem, we need to compute the differential of G at any given point φ_0 and expect the differential to be both injective and surjective.

This way, we get an isomorphism between two infinite-dimensional linear spaces. We use the differential rule for the determinant:

$$\begin{aligned} dG_{\varphi_0}(\varphi_0 + t\varphi) &= \frac{d}{dt}\bigg|_{t=0} G(\varphi_0 + t\varphi) \\ &= \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} \Delta_{\varphi_0}(\varphi_0). \end{aligned} \tag{3.8}$$

Thus, the differential of G at a point φ_0 is $\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} \Delta_{\varphi_0}$, where Δ_{φ_0} is the Laplacian in terms of the metric $\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}) dz^i \otimes d\bar{z}^j$.

Since B is an affine hyperplane, the tangent space of B at any point consists of the function in $C^{k-1,\alpha}(M)$ and satisfies $\int_M f = 0$, i.e.:m

$$\{f \in C^{k-1,\alpha}(M) \mid \int_M f = 0\}. \tag{3.9}$$

Next, we use the Fredholm alternative theory, [Eva10]. We know that the condition for the weak solution of $\Delta_{\varphi_0}\varphi = g$ to exist is $\int_M g dV_{\varphi_0} = 0$. Therefore, if we want the weak solution of the following equation exists:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} \Delta_{\varphi_0}\varphi = f, \tag{3.10}$$

which is equavalent to

$$\Delta_{\varphi_0}\varphi = \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right)^{-1} \det(g_{i\bar{j}}) f, \tag{3.11}$$

we need:

$$\int_M \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right)^{-1} \det(g_{i\bar{j}}) f dV_{\varphi_0} = \int_M f dV_{\varphi} = 0. \tag{3.12}$$

Thus, all the images of dG_{φ_0} belong to the tangent space of B. And now, we will try to demonstrate that the differential of G at some point φ_0, dG_{φ_0} is both injective and surjective:

By the classic Schauder estimates, we know that for the elliptic equation 3.10, we only need the function $f \in C^{k-1,\alpha}(M)$ to assure there is a solution $\varphi \in C^{k+1,\alpha}(M)$. This shows the differential map dG_{φ_0} is surjective.

We still need to prove dG_{φ_0} is injective. Consider the following equation:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} \Delta_{\varphi_0}\varphi = 0. \tag{3.13}$$

Apparently, $\varphi \equiv 0$ is a solution to the above equation. Because the difference of every two solutions to the equation 3.10 is a solution to the above equation. By the maximal principle, we also know that every two solutions to the above equation differ most by a constant. Since we already supposed φ to satisfy $\int_M \varphi = 0$, we know $\varphi \equiv 0$ is the unique solution to the equation 3.13.

Therefore, we proved that dG_{φ_0} is a linear, injective and surjective map. Apply the inverse function theorem. We know that dG_{φ_0} is invertible and G maps some open neighborhood of φ_0 to an open neighborhood of $G(\varphi_0)$ in B. Therefore, we know S is an open set.

Next, we need to prove S is a closed set:

We consider to prove S is self-sequentially compact, since in $\mathbb{R}$, a set is compact is equivalent to self-sequentially compact and also equivalent to a bounded closed set. Suppose $\{t_q\}$ is a sequence in S, i.e., we have the sequence $\{\varphi_q\} \subset C^{k+1,\alpha}(M)$, s.t.:

$$\det\left(g_{i\bar{j}} + \frac{\partial \varphi_q}{\partial z^i \partial \bar{z}^j}\right) \det\left(g_{i\bar{j}}\right)^{-1} = \mathrm{Vol}(M)\left(\int_M \exp\{t_q F\}\right)^{-1} \exp\{t_q F\}. \tag{3.14}$$

Since if we add or subtract any constant to the solution φ_q, we still get a solution to the above equation. W.l.o.g, we suppose $\int_M \varphi_q = 0$. Take differential w.r.t. $\frac{\partial}{\partial z^p}$ on both sides of the above equation in the normal coordinate and use the differential formula for the determinant. We have:

$$\begin{aligned}\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j}\right) \sum_{i,j} g_q'^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial \varphi_q}{\partial z^p}\right) \\ = \mathrm{Vol}(M)\left(\int_M \exp\{t_q F\}\right)^{-1} \frac{\partial}{\partial z^p}\left[\exp\{t_q F\} \det\left(g_{i\bar{j}}\right)\right],\end{aligned} \tag{3.15}$$

where $(g_q'^{i\bar{j}})$ represents for the inverse matrix of $(g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j})$.

We have proved there are uniform estimates for $1 + \varphi_{i\bar{i}}$, i.e., the above equation is uniformly elliptic, and together with the third order estimates for φ, we know the left-hand side is α-Hölder continuous, where α is an arbitrary number in $[0, 1]$.

Now, we use the regularity theory for uniform elliptic equations with continuous coefficients. By Schauder estimates, we will get the $C^{2,\alpha}(M)$ estimates of $\frac{\partial \varphi_q}{\partial z^p}$. Similarly, we have the $C^{2,\alpha}(M)$ estimates for $\frac{\partial \varphi_q}{\partial \bar{z}^p}$. Therefore, we know the coefficients of the above equation belong to $C^{1,\alpha}(M)$. We will finally get the $C^{k+1,\alpha}(M)$ estimates for φ_q using the so-called Bootstrap method. Then, by the Arzela-Ascoli theorem, we know there is a subsequence of $\{\varphi_q\}$ that converges to φ in the norm of $C^{k+1,\alpha}(M)$, and φ satisfies:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) \det(g_{i\bar{j}})^{-1} = \mathrm{Vol}(M)\left(\int_M \exp\{t_0 F\}\right)^{-1} \exp\{t_0 F\}, \tag{3.16}$$

where $t_0 = \lim_{q \to \infty} t_q$. Thus, a convergent subsequence of $\{t_q\}$ exists. And we know S is closed.

Therefore, we have the following theorem:

Theorem 2. *Let M be a compact Kähler manifold, and its Kähler metric is $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Suppose $F \in C^k(M), k \geq 3$ and $\int_M \exp F = Vol(M)$. Then, there is a function $\varphi \in C^{k+1,\alpha}(M), \forall 0 \leq \alpha < 1$, s.t. $\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}) dz^i \otimes d\bar{z}^j$ is still a Kähler metric, and satisfies:*

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) = \exp F \det\left(g_{i\bar{j}}\right). \tag{3.17}$$

Recall the Calabi Conjecture:

Let M be a compact Kähler manifold and its Kähler metric is $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$, its Ricci tensor is $\sum_{i,j} R_{i\bar{j}} dz^i \otimes d\bar{z}^j$, Shiing-Shen Chern [Che46] proved the $(1,1)$-form that relates to the Ricci tensor: $\frac{\sqrt{-1}}{2\pi} \sum_{i,j} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ represents the first Chern class of the manifold M.

Then, Calabi asked the following question:

If the $(1,1)$-form $\frac{\sqrt{-1}}{2\pi} \sum_{i,j} \tilde{R}_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is another representative element of the first Chern class. Can we find another Kähler metric, s.t. its Ricci tensor is exactly $\sum_{i,j} \tilde{R}_{i\bar{j}} dz^i \otimes d\bar{z}^j$

The above theorem confirms the Calabi Conjecture. In fact, we know:

$$R_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log \det(g_{i\bar{j}}). \tag{3.18}$$

By the assumption $\frac{\sqrt{-1}}{2\pi} \sum \tilde{R}_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ represents the first Chern class, and by the $\partial\bar{\partial}$-lemma, we have:

$$\tilde{R}_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}} - \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta}, \tag{3.19}$$

where f is a real-valued smooth function.

According to the above theorem, we can find a smooth function φ, s.t. $\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}) dz^i \otimes d\bar{z}^j$ is still a Kähler metric, and satisfies the equation:

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z^\alpha \partial \bar{z}^\beta}\right) \det(g_{\alpha\bar{\beta}})^{-1} = C \exp\{f\}, \tag{3.20}$$

where C is a fixed constant, s.t.:

$$\int_M C \exp\{f\} = \mathrm{Vol}(M). \tag{3.21}$$

Therefore, so far, we have proved the following theorem:

Theorem 3. *Let M be a compact Kähler manifold, and its Kähler metric is $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. If the $(1,1)$-form $\frac{\sqrt{-1}}{2\pi} \sum_{i,j} \tilde{R}_{i\bar{j}} dz^i \wedge d\bar{z}^j$ corresponds to the tensor $\sum_{i,j} \tilde{R}_{i\bar{j}} dz^i \otimes d\bar{z}^j$ represents the first Chern class $c_1(M)$. Then, we can find a Kähler metric $\sum_{i,j} \tilde{g}_{i\bar{j}} dz^i \otimes d\bar{z}^j$, s.t. its Ricci tensor equals to $\sum_{i,j} \tilde{R}_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Furthermore, the Kähler form of the the new metric is in the same Kähler class as the original Kähler form, and the Kähler metric satisfies the above conditions is unique.*

Chapter 4

Complex Monge-Ampère Equation with More General Right-Hand Side

In this chapter, we will study the following equation:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) = \exp\{F(x,\varphi)\}\det(g_{i\bar{j}}), \tag{4.1}$$

where $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$ is the Kähler metric of the compact Kähler manifold M, and F is a smooth real-valued function defined on $M \times \mathbb{R}$ satisfies that $\frac{\partial F}{\partial t} \geq 0$.

To solve the above equation, i.e. find such a real-valued smooth function φ, and s.t. $\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}) dz^i \otimes d\bar{z}^j$ is still a Kähler metric, we first integral on both sides of the above equation, to get a necessary condition. We need a smooth function ψ, s.t.:

$$\int_M \exp\{F(x,\psi)\} = \mathrm{Vol}(M). \tag{4.2}$$

By 2, we know if we take the right-hand side of 4.1 a fixed smooth function ψ, such a smooth solution φ exists. Next, we use an iteration method to solve this more general equation. We need a priori estimates for φ to ensure the solution is smooth.

4.1 A Priori C^0 Estimates of the Equation 4.1

We first prove the uniqueness of the solution of the equation 4.1:

Lemma 3. *Suppose φ and $\bar{\varphi}$ are two smooth solution of the equation 4.1, and satisfy:*

$$\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}) dz^i \otimes d\bar{z}^j, \tag{4.3}$$

and

$$\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2 \bar{\varphi}}{\partial z^i \partial \bar{z}^j})dz^i \otimes d\bar{z}^j, \tag{4.4}$$

are both the Kähler metric of the manifold M. Then, $\varphi - \bar{\varphi}$ is always a constant.

Proof. Since φ and $\bar{\varphi}$ both satisfy the equation 4.1, take quotient of the two equations of φ and $\bar{\varphi}$, and come up with the $\varphi - \bar{\varphi}$ term, we have:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \bar{\varphi}}{\partial z^i \partial \bar{z}^j} + \frac{\partial^2 (\varphi - \bar{\varphi})}{\partial z^i \partial \bar{z}^j}\right) \det\left(g_{i\bar{j}} + \frac{\partial^2 \bar{\varphi}}{\partial z^i \partial \bar{z}^j}\right)^{-1} = \exp\{F(x, \varphi) - F(x, \bar{\varphi})\}. \tag{4.5}$$

Suppose Δ_φ is the Laplacian in terms of the metric $\sum_{i,j}\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^i}\right) dz^i \otimes d\bar{z}^j$. We conveniently denote the above equation as:

$$\det\left(g_{i\bar{j}} + \bar{\varphi}_{i\bar{j}} + (\varphi - \bar{\varphi})_{i\bar{j}}\right) \det\left(g_{i\bar{j}} + \bar{\varphi}_{i\bar{j}}\right)^{-1} = \exp\{F(x, \varphi) - F(x, \bar{\varphi})\}. \tag{4.6}$$

We take the special coordinate, s.t. the matrix $(g_{i\bar{j}})$ to become the identity matrix, and $(\varphi_{i\bar{j}}), (\bar{\varphi}_{i\bar{j}})$ to be diagonal matrices. This can be achieved by using matrix transformation.

Then, in this coordinate, the above equation can be rewritten into:

$$\frac{\prod_{i=1}^m (1 + \bar{\varphi}_{i\bar{i}} + (\varphi - \bar{\varphi})_{i\bar{i}})}{\prod_{i=1}^m (1 + \bar{\varphi}_{i\bar{i}})} = \prod_{i=1}^m \left(1 + \frac{(\varphi - \bar{\varphi})_{i\bar{i}}}{1 + \bar{\varphi}_{i\bar{i}}}\right) = \exp\{F(x, \varphi) - F(x, \bar{\varphi})\}. \tag{4.7}$$

Use the mean value inequality, we have:

$$\begin{aligned} \frac{m + \Delta_\varphi(\varphi - \bar{\varphi})}{m} = \frac{\sum_{i=1}^m \left(1 + \frac{(\varphi - \bar{\varphi})_{i\bar{i}}}{1 + \varphi_{i\bar{i}}}\right)}{m} &\geq \sqrt[m]{\prod_{i=1}^m \left(1 + \frac{(\varphi - \bar{\varphi})_{i\bar{i}}}{1 + \varphi_{i\bar{i}}}\right)} \\ &= \exp\{\frac{1}{m}\left[F(x, \varphi) - F(x, \bar{\varphi})\right]\}, \end{aligned} \tag{4.8}$$

which is equivalent to:

$$m + \Delta_\varphi(\varphi - \bar{\varphi}) \geq m \exp\{\frac{1}{m}[F(x, \varphi) - F(x, \bar{\varphi})]\}. \tag{4.9}$$

Using the equation, we get the following inequality:

$$\frac{m + \Delta_\varphi(\varphi - \bar{\varphi})}{m} \geq \left[\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right) \det\left(g_{i\bar{j}} + \bar{\varphi}_{i\bar{j}}\right)^{-1}\right]^{1/m} \tag{4.10}$$

By the classic mean value theorem, we know:

$$F(x, \varphi) - F(x, \bar{\varphi}) = \int_{\bar{\varphi}(x)}^{\varphi(x)} \frac{\partial F}{\partial t}(x, \tau) d\tau = \frac{\partial F}{\partial t}(x, \bar{t}(x))(\varphi(x) - \bar{\varphi}(x)), \tag{4.11}$$

where $\bar{t}(x)$ is a number between $\inf(\varphi(x), \bar{\varphi}(x))$ and $\sup(\varphi(x), \bar{\varphi}(x))$.

Since $\frac{\partial F}{\partial t} \geq 0$, from 4.11, we know that when $\varphi(x) - \bar{\varphi}(x) > 0$, we have:

$$F(x, \varphi) - F(x, \bar{\varphi}) \geq 0. \tag{4.12}$$

Thus, the right-hand side of 4.9 becomes $m \exp\{\frac{1}{m}[F(x, \varphi) - F(x, \bar{\varphi})]\} \geq m$, which shows $\Delta_{\varphi}(\varphi - \bar{\varphi}) \geq 0$.

Then, by the maximal principle, we know $\varphi - \bar{\varphi}$ is a constant: Consider the following integral on the set $E = \{x \mid \varphi(x) - \bar{\varphi}(x) < 0\}$:

$$0 \geq \int_E (\varphi - \bar{\varphi}) \cdot \Delta_{\varphi}(\varphi - \bar{\varphi}) = \int_E |\nabla(\varphi - \bar{\varphi})|^2 \geq 0. \tag{4.13}$$

Thus, we have $\varphi - \bar{\varphi}$ is a constant on E. Similarly, exchange the position of $\varphi, \bar{\varphi}$, and we will know $\varphi - \bar{\varphi}$ is also equal to some constant on $M \backslash E$. Also, because $\varphi - \bar{\varphi}$ is a smooth function, we know that on the whole manifold M, $\varphi - \bar{\varphi}$ is always a constant. □

Next, we will use iteration to prove the existence of the solution of 4.1 and its C^0 estimates. By the theorem 2, there is a smooth function φ_0, s.t. $\sum_{i,j}(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}) dz^i \otimes d\bar{z}^j$ is a Kähler metric and satisfies the equation:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\right) = \exp\{F(x, \psi)\} \det(g_{i\bar{j}}). \tag{4.14}$$

Set $\bar{\varphi}_0 = \varphi_0 + \sup_M |\varphi_0 - \psi|$ and $\bar{\bar{\varphi}}_0 = \varphi_0 - \sup_M |\varphi_0 - \psi|$. They are all add or subtract a constant of φ_0. Thus, they both satisfy the above equation. Further, by definition, for any $x \in M$, we have:

$$\bar{\varphi}_0(x) \geq \psi(x) \geq \bar{\bar{\varphi}}_0(x). \tag{4.15}$$

Set $A := \{(x, t) | x \in M, \bar{\varphi}_0(x) \geq t \geq \bar{\bar{\varphi}}_0(x)\}$. Since M is a compact manifold, and $\bar{\varphi}_0, \bar{\bar{\varphi}}_0$ are smooth functions on the compact manifold, we know they are both bounded. Thus, A is a compact subset of $M \times \mathbb{R}$, and we can define:

$$k := \sup_{(x,t) \in A} \frac{\partial F}{\partial t}(x, t) + 1 > 0. \tag{4.16}$$

We will then use the following iteration scheme:

To make a difference in the indices, we use i to denote the iteration indices and α, β to denote the tensor indices. Set $\bar{\varphi}_i, \bar{\bar{\varphi}}_i$ are smooth solutions of the following equation, respectively:

$$\begin{aligned} \det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta}\right) &= \exp\left\{k\bar{\varphi}_i + F\left(x, \bar{\varphi}_{i-1}\right) - k\bar{\varphi}_{i-1}\right\} \det(g_{\alpha\bar{\beta}}), \\ \det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\bar{\varphi}}_i}{\partial z^\alpha \partial \bar{z}^\beta}\right) &= \exp\left\{k\bar{\bar{\varphi}}_i + F\left(x, \bar{\bar{\varphi}}_{i-1}\right) - k\bar{\bar{\varphi}}_{i-1}\right\} \det(g_{\alpha\bar{\beta}}), \end{aligned} \tag{4.17}$$

where $\sum_{\alpha,\beta}\left(g_{\alpha\bar{\beta}}+\frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha\partial\bar{z}^\beta}\right)dz^\alpha\otimes d\bar{z}^\beta$ and $\sum_{\alpha,\beta}\left(g_{\alpha\bar{\beta}}+\frac{\partial^2\bar{\bar{\varphi}}_i}{\partial z^\alpha\partial\bar{z}^\beta}\right)dz^\alpha\otimes d\bar{z}^\beta$ are both Kähler metric.

It's natural to ask about the existence and uniqueness of the solution if we wish to use the above iteration scheme. We have the following lemma to ensure we can make the above assumptions.

Lemma 4. *Suppose M is a compact Kähler manifold, and its Kähler metric is $\sum_{i,j}(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j})dz^i\otimes d\bar{z}^j$. And F is an arbitrary smooth function on M. Then, for any constant $k>0$, there is a unique smooth solution to the equation:*

$$\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)=\exp\{k\varphi+F\}\det(g_{i\bar{j}}), \tag{4.18}$$

s.t. $\sum_{i,j}(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j})dz^i\otimes d\bar{z}^j$ is again a Kähler metric.

Proof. Following the proof of the theorem 2, we still use the continuity method to show the existence. Consider a family of equations parametrized by t:

$$\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)=\exp\{k\varphi+tF\}\det(g_{i\bar{j}}). \tag{4.19}$$

The most crucial step is the C^0 estimate of φ, but the condition $k>0$ makes this case much easier:

Suppose the maximal point of φ on the manifold M is x_0. At this point, the Hessian matrix of φ is non-positive definite. Thus, we can choose a special coordinate, s.t., for any $\forall i, \varphi_{i\bar{i}}\leq 0$. And this leads to

$$\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\leq 1. \tag{4.20}$$

By 4.19, we know the above inequality equals to $\exp\{k\varphi(x_0)+tF(x_0)\}\leq 1$, i.e. $k\varphi(x_0)+tF(x_0)\leq 0$. Shift terms, we have $\sup_M\varphi=\varphi(x_0)\leq-\frac{t}{k}F(x_0)$, and this gives us the estimate of $\sup_M\varphi$.

Similarly, suppose the minimal point of φ is x_1, we have:

$$\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\geq 1. \tag{4.21}$$

Thus, we have $\inf_M\varphi\geq-\frac{t}{k}F(x_1)$. Therefore, we have the C^0 estimates of φ. Then, following the proof of theorem 2, we have the existence of the solution.

Also, since $\frac{\partial(k\varphi+tF)}{\partial\varphi}=k>0$, we know the equation 4.18 satisfies the condition in 3. Thus, the difference between the two solutions is, at most, a constant. Suppose $\varphi-\tilde{\varphi}=c$, by equation 4.18, the quotient of two equations gives us $e^{kc}=1$. Because $k\neq 0$, we know $c=0$. Therefore, the solution of the above equation is also unique. $\square$

The above theorem assures the existence and uniqueness of the solution. Next,we will prove the solutions satisfy the equation 4.17, also satisfy the following inequality, for any $i \geq 0$

$$\bar{\bar{\varphi}}_0 \leq \bar{\bar{\varphi}}_i \leq \bar{\bar{\varphi}}_{i+1} \leq \bar{\varphi}_{i+1} \leq \bar{\varphi}_i \leq \bar{\varphi}_0. \tag{4.22}$$

We prove this by induction: When $i = 0$, suppose $\bar{\varphi}_1$ safisfies the equation 4.17, and we use the inequality 4.15 to scale it. Since φ_0 satisfies the equation 4.14, we have the following inequality:

$$\begin{aligned}
\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\varphi}_1}{\partial z^\alpha \partial \bar{z}^\beta}\right) &= \exp\left\{k\bar{\varphi}_1 - k\bar{\varphi}_0\right\} \exp\left\{F\left(x, \bar{\varphi}_0\right)\right\} \det\left(g_{\alpha\bar{\beta}}\right) \\
&\geq \exp\left\{k\left(\bar{\varphi}_1 - \bar{\varphi}_0\right)\right\} \exp\{F(x,\psi)\} \det\left(g_{\alpha\bar{\beta}}\right) \\
&= \exp\left\{k\left(\bar{\varphi}_1 - \bar{\varphi}_0\right)\right\} \det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\varphi}_0}{\partial z^\alpha \partial \bar{z}^\beta}\right).
\end{aligned} \tag{4.23}$$

At the maximal point of $\bar{\varphi}_1 - \bar{\varphi}_0$, we have $\Delta(\bar{\varphi}_1 - \bar{\varphi}_0) \leq 0$. Thus, we can rewrite the above inequality:

$$\frac{1}{m}[m + \Delta(\bar{\varphi}_1 - \bar{\varphi}_0)] \leq 1. \tag{4.24}$$

By a completely same computation with the left-hand side of 4.10, in the special coordinate, we know that the above inequality is equivalent to:

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\varphi}_1}{\partial z^\alpha \partial \bar{z}^\beta}\right) \leq \det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\varphi}_0}{\partial z^\alpha \partial \bar{z}^\beta}\right). \tag{4.25}$$

Together with 4.23, we know $\sup_M(\bar{\varphi}_1 - \bar{\varphi}_0) \leq 0$

By a similar discussion, write out the equation that $\bar{\bar{\varphi}}_1$ satisfied, and use the inequality 4.15, we have $\sup_M(\bar{\bar{\varphi}}_0 - \bar{\bar{\varphi}}_1) \leq 0$. This proves the second and the fourth inequalities in 4.22 hold for $i = 0$.

We still need to prove the third inequality holds, i.e., $\bar{\bar{\varphi}}_1 \leq \bar{\varphi}_1$: In the equation system 4.17, set $i = 1$, and take quotient, we have:

$$\begin{aligned}
\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\varphi}_1}{\partial z^\alpha \bar{z}^\beta}\right) &\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\bar{\varphi}}_1}{\partial z^\alpha \partial \bar{z}^\beta}\right)^{-1} \\
&= \exp\left\{k\bar{\varphi}_1 - k\bar{\bar{\varphi}}_1 + F\left(x, \bar{\varphi}_0\right) - F\left(x, \bar{\bar{\varphi}}_0\right) - k\left(\bar{\varphi}_0 - \bar{\bar{\varphi}}_0\right)\right\}.
\end{aligned} \tag{4.26}$$

By the definition of k in 4.16, and $\bar{\varphi}_0 \geq \bar{\bar{\varphi}}_0$, together with the differential mean value theorem 4.11, we know for any $x \in M$, we have:

$$F\left(x, \bar{\varphi}_0\right) - F\left(x, \bar{\bar{\varphi}}_0\right) - k\left(\bar{\varphi}_0 - \bar{\bar{\varphi}}_0\right) \leq 0. \tag{4.27}$$

Thus, back to the equation 4.26, we have:

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\bar{\varphi}}_1}{\partial z^\alpha \partial \bar{z}^\beta}\right) \det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \bar{\varphi}_1}{\partial z^\alpha \partial \bar{z}^\beta}\right)^{-1} \geq \exp\left\{k\left(\bar{\bar{\varphi}}_1 - \bar{\varphi}_1\right)\right\}. \tag{4.28}$$

At the maximal point of $\bar{\bar{\varphi}}_1 - \bar{\varphi}_1$, we have $\Delta(\bar{\bar{\varphi}}_1 - \bar{\varphi}_1) \leq 0$ which is equivalent to $\frac{1}{m}[m + \Delta(\bar{\varphi}_1 - \bar{\varphi}_0)] \leq 1$. By the inequality 4.10, we know the left-hand side of the above inequality is less or equal to 1, together with $k \geq 0$, we have $\bar{\varphi}_1 \geq \bar{\bar{\varphi}}_1$.

Next, we use the induction. Suppose for any $j < i$, we have:

$$\bar{\bar{\varphi}}_0 \leq \bar{\bar{\varphi}}_{j-1} \leq \bar{\bar{\varphi}}_j \leq \bar{\varphi}_j \leq \bar{\varphi}_{j-1} \leq \bar{\varphi}_0. \tag{4.29}$$

We will prove that the conclusion still holds if we replace the j in the above inequality with i. We set $\bar{\varphi}_i$ and $\bar{\varphi}_{i-1}$ as the solutions in 4.17, and take quotient:

$$\begin{aligned} &\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta}\right)\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\bar{\varphi}_{i-1}}{\partial z^\alpha \partial \bar{z}^\beta}\right)^{-1} \\ &\quad = \exp\left\{k\bar{\varphi}_i - k\bar{\varphi}_{i-1} + F\left(x, \bar{\varphi}_{i-1}\right) - F\left(x, \bar{\varphi}_{i-2}\right) - k\left(\bar{\varphi}_{i-1} - \bar{\varphi}_{i-2}\right)\right\}. \end{aligned} \tag{4.30}$$

By the induction hypothesis, we have $\bar{\varphi}_0 \geq \bar{\varphi}_{i-2} \geq \bar{\varphi}_{i-1} \geq \bar{\bar{\varphi}}_0$, together with the fact that F is non-decreasing w.r.t. the second variable. Use a similar argument in 4.28 to get:

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta}\right)\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\bar{\varphi}_{i-1}}{\partial z^\alpha \partial \bar{z}^\beta}\right)^{-1} \geq \exp\left\{k\bar{\varphi}_i - k\bar{\varphi}_{i-1}\right\}. \tag{4.31}$$

Thus, we know $\bar{\varphi}_{i-1} \geq \bar{\varphi}_i$. By a similar argument, we will also have $\bar{\bar{\varphi}}_{i-1} \leq \bar{\bar{\varphi}}_i$.

Now, we still need to prove the third inequality $\bar{\varphi}_i \geq \bar{\bar{\varphi}}_i$ in 4.22.

Set $\bar{\varphi}_i, \bar{\bar{\varphi}}_i$ as solutions in 4.17, and take quotient:

$$\begin{aligned} &\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta}\right)\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\bar{\bar{\varphi}}_i}{\partial z^\alpha \partial \bar{z}^\beta}\right) \\ &\quad = \exp\left\{k\bar{\varphi}_i - k\bar{\bar{\varphi}}_i + F\left(x, \bar{\varphi}_{i-1}\right) - F\left(x, \bar{\bar{\varphi}}_{i-1}\right) - k\bar{\varphi}_{i-1} + k\bar{\bar{\varphi}}_{i-1}\right\}. \end{aligned} \tag{4.32}$$

Since we have $\bar{\varphi}_i \geq \bar{\bar{\varphi}}_{i-1}$, use the induction hypothesis repeatedly, we have $\bar{\varphi}_i \geq \bar{\bar{\varphi}}_i$.

Therefore, for any positive integer i, the inequality 4.22 holds, and $\bar{\varphi}_i, \bar{\bar{\varphi}}_i$ are uniformly bounded. This gives the C^0 estimates for the solution of 4.17.

By the structure of the equation, together with the squeeze rule, we can see if we let i go to infinity, the solution will converge to a unique limit function φ. Thus, the equation will converge to 4.1. In this way, we have the C^0 estimates of the wanted more general complex Monge-Ampère equation.

4.2 A Priori C^2 Estimates of the Equation 4.1

Next, we will compute the C^2 estimates of the solution $\bar{\varphi}_i$ of the equation 4.17. The C^2 estimates of $\bar{\bar{\varphi}}_i$ is similar. We want to show the uniform estimates of $\frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta}$. We only need to show the estimate of $\sup_M \Delta\bar{\varphi}_i$ by the equation and the mean value inequality.

Similar to the C^2 estimates in the Chapter 1. The main difference now is the right-hand side term is more complicated than $\exp\{F\}$, but the computation is quite similar. And now we have the C^0 estimates of the solution.

Let Δ_i be the Laplacian of the metric $\sum_{\alpha,\beta}\left(g_{\alpha\bar{\beta}}+\frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha\partial\bar{z}^\beta}\right)dz^\alpha\otimes d\bar{z}^\beta$. Suppose C is a positive constant, s.t. $C+\inf_{i\neq l}(R_{i\bar{i}l\bar{l}})>1$. To show the second order estimates, we still consider the function $\Delta_i[\exp\{-C\bar{\varphi}_i\}(m+\Delta\bar{\varphi}_i)]$. With a similar computation in 2.34:

$$\begin{aligned}\Delta_i[\exp\{-C\bar{\varphi}_i\}(m+\Delta\bar{\varphi}_i)]=\exp\{-C\bar{\varphi}_i\}&\Bigg[k(\Delta\bar{\varphi}_i-\Delta\bar{\varphi}_{i-1})+\sum_{\alpha,\beta}g^{\alpha\bar{\beta}}\frac{\partial F(x,\bar{\varphi}_{i-1})}{\partial z^\alpha\partial\bar{z}^\beta}\\&+\sum_{\alpha,\beta}g^{\alpha\bar{\beta}}\frac{\partial^2F(x,\bar{\varphi}_{i-1})}{\partial t\partial z^\alpha}\frac{\partial\bar{\varphi}_{i-1}}{\partial\bar{z}^\beta}+\sum_{\alpha,\beta}g^{\alpha\bar{\beta}}\frac{\partial^2F(x,\bar{\varphi}_{i-1})}{\partial t\partial\bar{z}^\beta}\frac{\partial\bar{\varphi}_{i-1}}{\partial z^\alpha}\\&+\frac{\partial^2F}{\partial t^2}(x,\bar{\varphi}_{i-1})\sum_{\alpha,\beta}g^{\alpha\bar{\beta}}\frac{\partial\bar{\varphi}_{i-1}}{\partial z^\alpha}\frac{\partial\bar{\varphi}_{i-1}}{\partial\bar{z}^\beta}\\&+\frac{\partial F}{\partial t}(x,\bar{\varphi}_{i-1})\Delta\bar{\varphi}_{i-1}-m^2\inf_{i\neq l}R_{i\bar{i}l\bar{l}}\Bigg]-C\exp\{-C\bar{\varphi}_i\}m(m+\Delta\bar{\varphi}_i)\\&+\left(C+\inf_{i\neq l}(R_{i\bar{i}l\bar{l}})\right)\exp\{-C\bar{\varphi}_i\}(m+\Delta\bar{\varphi}_i)\left(\sum_\alpha\frac{1}{1+(\bar{\varphi}_i)_{\alpha\bar{\alpha}}}\right).\end{aligned}\tag{4.33}$$

The terms in the square brackets come from computing the Laplacian w.r.t. the initial metric $\sum_{\alpha,\beta}g_{\alpha\bar{\beta}}dz^\alpha\otimes d\bar{z}^\beta$ of the composite function $F(x,\varphi(x))$.

We have already estimated $\sup_M|\bar{\varphi}_i|$, and we know the $L^1(M)$ estimates of $\bar{\varphi}_i$ only rely on the maximum of the function $\bar{\varphi}_i$ and the Green function of the compact manifold M. Thus, follow the inequality 2.52 in the Chapter 1, we have the $L^1(M)$ estimates for $\bar{\varphi}_i$. Then, by the Schauder estimates, there is a constant C_1 that is independent of i, s.t.:

$$\sup_M|\nabla\bar{\varphi}_i|\leq C_1\left(\sup_M|\Delta\bar{\varphi}_i|+1\right).\tag{4.34}$$

Similar to 2.39, we know there is a constant C_2 that is independent of i, s.t.:

$$\begin{aligned}\sum_\alpha\frac{1}{1+(\bar{\varphi}_i)_{\alpha\bar{\alpha}}}&\geq(m+\Delta\bar{\varphi}_i)^{1/(m-1)}\left(\exp\left\{\frac{-k}{m-1}(\bar{\varphi}_i-\bar{\varphi}_{i-1})+\frac{1}{m-1}F(x,\bar{\varphi}_{i-1})\right\}\right)\\&\geq C_2(m+\Delta\bar{\varphi}_i)^{1/(m-1)}.\end{aligned}\tag{4.35}$$

The second line of the above inequality is because we have the C^0 estimates for $\bar{\varphi}_i$. Thus, F only takes the value on a compact set, and we know F is bounded.

With the above computation, substitute the above inequality, 4.34, and the C^0 estimates into 4.33. We mainly focus on the second-order terms since other lower terms are already estimated. In particular, consider the third line of 4.33, we have:

$$|\nabla\varphi|^2 \geq -|\nabla\varphi| \geq -\sup_M |\nabla\varphi| \geq -C_1\left(\sup_M |\Delta\bar{\varphi}_i| + 1\right). \tag{4.36}$$

Therefore, we know there are positive constants C_3, C_4, C_5, C_6 that not depend on i, s.t.:

$$\begin{aligned}\Delta_i\left[\exp\left\{-C\bar{\varphi}_i\right\}(m+\ \Delta\bar{\varphi}_i)\right] \geq & -C_3\left(m+\sup_M \Delta\bar{\varphi}_i\right) - C_4\left(m+\sup_M \Delta\bar{\varphi}_{i-1}\right) \\ & + C_5\left(m+\sup_M \Delta\bar{\varphi}_i\right)^{1+1/(m-1)} - C_6.\end{aligned} \tag{4.37}$$

At the maximal point of $\exp\left\{-C\bar{\varphi}_i\right\}(m+\Delta\bar{\varphi}_i)$, the left-hand side of the above inequality is less or equal to 0. Thus, we have:

$$C_5\left(m+\sup_M \Delta\bar{\varphi}_i\right)^{1+1/(m-1)} \leq C_3\left(m+\sup_M \Delta\bar{\varphi}_i\right) + C_4\left(m+\sup_M \Delta\bar{\varphi}_{i-1}\right) + C_6. \tag{4.38}$$

We know that for any $x \geq 0$, we can always find large $a, b > 0$, s.t.:

$$x^{1+1/(m-1)} \geq ax - b. \tag{4.39}$$

Thus, we can always find positive constants C_7, C_8, s.t. for any $x \geq 0$, we have:

$$\left(m+\sup_M \Delta\bar{\varphi}_i\right)^{1+1/(m-1)} \geq C_7\left(m+\sup_M \Delta\bar{\varphi}_i\right) - C_8. \tag{4.40}$$

Substitute the inequality 4.40 into 4.38:

$$C_5C_7\left(m+\sup_M \Delta\bar{\varphi}_i\right) - C_5C_8 \leq C_3\left(m+\sup_M \Delta\bar{\varphi}_i\right) + C_4\left(m+\sup_M \Delta\bar{\varphi}_{i-1}\right) + C_6. \tag{4.41}$$

Shift terms, and get:

$$(C_5C_7 - C_3)\left(m+\sup_M \Delta\bar{\varphi}_i\right) \leq C_4\left(m+\sup_M \Delta\bar{\varphi}_{i-1}\right) + C_6 + C_5C_8. \tag{4.42}$$

We can take C_7 large enough, s.t. we can divide $C_5C_7 - C_3$ on both sides of the above inequality. As a result, the coefficient of the first term on the right-hand side is smaller than $\frac{1}{2}$. Thus, there is a positive constant $\tilde{C}$ that is independent of i, s.t.:

$$\left(m+\sup_M \Delta\bar{\varphi}_i\right) \leq \frac{1}{2}\left(m+\sup_M \Delta\bar{\varphi}_{i-1}\right) + \tilde{C}. \tag{4.43}$$

With the above inequality, we can iterate it repeatedly to get the upper bound estimates of $\Delta\bar{\varphi}_i$:

$$m+\sup_M \Delta\bar{\varphi}_i \leq \left(\frac{1}{2}\right)^i\left(m+\sup_M \Delta\bar{\varphi}_0\right) + 2\tilde{C}. \tag{4.44}$$

According to the theorem 2, we know the right-hand side above is bounded. And by the complex Monge-Ampère equation, we get uniform second-order estimates of $\bar{\varphi}_i$.

4.3 A Priori C^3 Estimates of the Equation 4.1

Next, we give the C^3 estimates, i.e. the uniform estimates for $\frac{\partial^3\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta \partial z^\gamma}$: Similar to the Chapter 1, define:

$$S_i := \sum g_i'^{\alpha\bar{l}} g_i'^{\bar{\beta}p} g_i'^{\gamma\bar{q}} \frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta \partial z^\gamma} \frac{\partial^2\bar{\varphi}_i}{\partial \bar{z}^l \partial z^p \partial \bar{z}^q}, \tag{4.45}$$

where the matrix $(g_i'^{\alpha\bar{l}})$ is the inverse matrix of $\left(g_{\alpha\bar{l}} + \frac{\partial^2\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^l}\right)$. By a similar computation to 2.128, we have:

$$\Delta_i \left(S_i + C_9 \Delta\bar{\varphi}_i\right) \geq C_{10} S_i - C_{11}\sqrt{S_i}\sqrt{S_{i-1}} - C_{12}, \tag{4.46}$$

where $C_9, C_{10}, C_{11}, C_{12}$ are all positive constants that is independent on i.

We have the various estimates for $|\Delta\bar{\varphi}_i|$, now by the maximal principle:

$$\sup_M S_i \leq \frac{C_{11}}{C_{10}} \sqrt{\sup_M S_i} \sqrt{\sup_M S_{i-1}} + \frac{C_{12}}{C_{10}} + C_9 \sup_M \left|\Delta\bar{\varphi}_i\right|. \tag{4.47}$$

Since we can adjust C_9, C_{12}, s.t. C_{10} to be large in 4.46, w.l.o.g. we suppose $\frac{C_{11}}{C_{10}} \leq \frac{1}{2}$. By the above inequality:

$$\sup_M S_i \leq \frac{1}{3} \sup_M S_{i-1} + \frac{4C_{12}}{C_{10}} + 4C_9 \sup_M \left|\Delta\bar{\varphi}_i\right|. \tag{4.48}$$

Continuing to iterate the above inequality, we have the uniform estimates for S_i. Thus, we have the uniform estimates for $\frac{\partial^3\bar{\varphi}_i}{\partial z^\alpha \partial \bar{z}^\beta \partial z^\gamma}$.

With these estimates for $\bar{\varphi}_i$, let i go to infinity, we get a solution φ to the equation 4.1, and the Schauder theory assures φ to be smooth.

Therefore, we have the following theorem:

Theorem 4. *Set M to be a Kähler manifold with Kähler metric $\sum_{i,\bar{j}} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Suppose F is a smooth function defined on $M \times \mathbb{R}$, and satisfies $\frac{\partial F}{\partial t} \geq 0$. Suppose we know in advance there is a real-valued smooth function ψ defined on M, s.t. $\int_M \exp\{F(x, \psi(x))\} = \mathrm{Vol}(M)$.*

Then, there is a real-valued smooth function φ on M satisfies the equation:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) = \exp\{F(x, \varphi\} \det(g_{i\bar{j}}), \tag{4.49}$$

and $\sum_{i,j}\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) dz^i \otimes d\bar{z}^j$ is still a Kähler metric. Furthermore, if two smooth functions both satisfy the above conditions, they must differ by a constant, at most.

4.4 The Existence and Uniqueness of the Kähler-Einstein Metric When $c_1(M) \leq 0$

Next, we want to show that, with some conditions, there is a uniqueness Kähler-Einstein metric on a compact Kähler manifold.

Definition 2. *The Kähler metric ω of a Kähler manifold (M, ω, J) is called a Kähler-Einstein metric if there is a constant λ, s.t.:*

$$Ric(\omega) = \lambda\omega. \tag{4.50}$$

Because the Ricci form $Ric(\omega) := \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$ is rescale invariant, we can always suppose λ to be $-1, 0, 1$.

Since $\frac{1}{2\pi}\sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is a representative element of the first Chern class, when there is a sign for the first Chern class, we should try to find the λ with the same sign. For example, when the first Chern class is negative definite $c_1(M) < 0$, i.e., there is a representative element whose coefficient matrix is negative definite, we should try to find the Kähler-Einstein metric ω, s.t. $Ric(\omega) = -\omega$.

In particular, we call the Kähler manifold with negative first Chern class a General Type. When the first Chern class equals 0, and if the metric is Kähler-Einstein, we have $Ric(\omega) = 0$, i.e., the Ricci curvature tensor is everywhere equals to 0, we call this manifold with such a metric as Ricci-flat manifold, also known as the Calabi-Yau manifold. We call the Kähler manifold with positive first Chern class the Fano manifold.

When $c_1(M) = 0$, we want to find the metric s.t. its Ricci curvature always equals 0. We immediately know the existence and uniqueness by the theorem 3. Next, we consider the case that $c_1(M) < 0$:

We need to use the theorem we just proved 4.

We know that $-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(g_{i\bar{j}})$ is a representative element of the first Chern class $c_1(M)$ of the Kähler manifold (M, ω). Since $c_1(M) < 0$, i.e., $-c_1(M) > 0$, we know there is a positive definite $(1,1)$-form that represents the negative first Chern class, denoted as $\frac{\sqrt{-1}}{2\pi}\sum_{i,j} g_{i\bar{j}}dz^i \wedge d\bar{z}^j$. That's to say $-\frac{\sqrt{-1}}{2\pi}\sum_{i,j} g_{i\bar{j}}dz^i \wedge d\bar{z}^j$ represents $c_1(M)$. According to the $\partial\bar{\partial}$-lemma, there is a smooth real-valued function f, s.t.:

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(g_{i\bar{j}}) + \frac{\sqrt{-1}}{2\pi}\sum_{i,j} g_{i\bar{j}}dz^i \wedge d\bar{z}^j = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f. \tag{4.51}$$

This equality is equivalent to

$$-\partial\bar{\partial}\log\det(g_{i\bar{j}}) = -\sum_{i,j} g_{i\bar{j}}dz^i \wedge d\bar{z}^j - \partial\bar{\partial}f. \tag{4.52}$$

In the theorem 4, we consider a special form of the equation, take $F(x, \varphi) = \varphi - f$:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial\bar{z}^j}\right) = \exp\{\varphi - f\}\det(g_{i\bar{j}}). \tag{4.53}$$

Take $-\partial\bar{\partial}\log$ on both sides of the equation:

$$-\partial\bar{\partial}\log\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)=-\partial\bar{\partial}\varphi+\partial\bar{\partial}f-\partial\bar{\partial}\log\det(g_{i\bar{j}}). \tag{4.54}$$

By the theorem 4, we know there is a smooth solution φ, s.t. $\sum_{i,j}\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)dz^i\otimes d\bar{z}^j$ is still a Kähler metric. Combine the above two equality 4.52 and 4.54, we have:

$$\begin{aligned}-\partial\bar{\partial}\log\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^i}\right)&=-\partial\bar{\partial}\varphi+\partial\bar{\partial}f-\sum_{i,j}g_{i\bar{j}}dz^i\wedge d\bar{z}^j-\partial\bar{\partial}f\\&=-\sum_{i,j}\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)dz^i\wedge d\bar{z}^j.\end{aligned} \tag{4.55}$$

i.e.

$$Ric_{\omega_\varphi}=-\sqrt{-1}\partial\bar{\partial}\log\det\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^i}\right)=-\sqrt{-1}\sum_{i,j}\left(g_{i\bar{j}}+\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)dz^i\wedge d\bar{z}^j=-\omega_\varphi. \tag{4.56}$$

Therefore, we find the wanted Kähler-Einstein metric.

Next, we prove the uniqueness:

Suppose $\sum_{i,j}g_{i\bar{j}}dz^i\otimes d\bar{z}^j$ is a Kähler-Einstein metric and $\sum_{i,j}\tilde{g}_{i\bar{j}}dz^i\otimes d\bar{z}^j$ is another Kähler-Einstein metric. Since $Ric_{\omega_{\tilde{g}}}=-\omega_{\tilde{g}}$ and the left-hand side is a representative element of the first Chern class, we know $\omega_{\tilde{g}}$ also represents the first Chern class. Thus, we can find a smooth function ψ, s.t. $\tilde{g}_{i\bar{j}}=g_{i\bar{j}}+\frac{\partial^2\psi}{\partial z^i\partial\bar{z}^j}$. On the other hand, by the definition of Kähler-Einstein metric, we have:

$$\begin{aligned}-\partial\bar{\partial}\log\det\left(g_{i\bar{j}}+\frac{\partial^2\psi}{\partial z^i\partial\bar{z}^i}\right)&=-\sum_{i,j}g_{i\bar{j}}dz^i\wedge d\bar{z}^j-\partial\bar{\partial}\psi\\&=-\partial\bar{\partial}\log\det(g_{i\bar{j}})+\partial\bar{\partial}f-\partial\bar{\partial}\psi.\end{aligned} \tag{4.57}$$

Shift terms, we have:

$$\partial\bar{\partial}\log\left[\det\left(g_{i\bar{j}}+\frac{\partial^2\psi}{\partial z^i\partial\bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\exp\{-\psi+f\}\right]=0. \tag{4.58}$$

i.e.

$$\log\left[\det\left(g_{i\bar{j}}+\frac{\partial^2\psi}{\partial z^i\partial\bar{z}^j}\right)\det(g_{i\bar{j}})^{-1}\exp\{-\psi+f\}\right] \tag{4.59}$$

is a pluriharmonic function, and by integrating by parts, we know that such functions on a closed manifold must be equal to constants. Thus, the above term equals some constant c, and we have:

$$\det\left(g_{i\bar{j}}+\frac{\partial^2\psi}{\partial z^i\partial\bar{z}^j}\right)=\exp\{\psi+c-f\}\det(g_{i\bar{j}}). \tag{4.60}$$

By the lemma 3, we know $\varphi - \psi$ is a constant. Thus, we have:

$$\sum_{i,j} \tilde{g}_{ij} dz^i \otimes d\bar{z}^j = \sum_{ij} \left(g_{ij} + \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} \right) dz^i \otimes d\bar{z}^j = \sum_{i,j} \left(g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^i} \right) dz^i \otimes d\bar{z}^j. \tag{4.61}$$

This shows the uniqueness of the Kähler-Einstein metric.

To sum up, we have the following theorem:

Theorem 5. *Let (M, ω, J) be a compact Kähler manifold with $c_1(M) \leq 0$. Then, there is a unique Kähler-Einstein metric.*

Chapter 5

Further Development

The proof of the Calabi conjecture profoundly impacts geometry, algebra, and even theoretical physics. It also created the frame to solve geometric problems with analytic tools and also, in turn, facilitated the study of equations.

After solving the existence and uniqueness of the Kähler-Einstein metric on the compact Kähler manifold with positive definite first Chern class, many new problems, concepts, and methods are prompted and studied. For example, the existence of the Kähler-Einstein metric on the Fano manifold closely relates to the concept of K-stability, which is widely studied in algebraic geometry. Also, Calabi-Yau manifolds are important in mathematical physics. Furthermore, people also do related research on the complete but non-compact manifolds.

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